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Random walk in a high density dynamic random environment

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Abstract

The goal of this note is to prove a law of large numbers for the empirical speed of a green particle that performs a random walk on top of a field of red particles which themselves perform independent simple random walks on $\mathbb{Z}^d$, $d \geq 1$. The red particles jump at rate 1 and are in a Poisson equilibrium with density μ. The green particle also jumps at rate 1, but uses different transition kernels p' and p'' depending on whether it sees a red particle or not. It is shown that, in the limit as $\mu \to \infty$, the speed of the green particle tends to the average jump under p'. This result is far from surprising, but it is non-trivial to prove. The proof that is given in this note is based on techniques that were developed in Kesten and Sidoravicius (2005) to deal with spread-of-infection models. The main difficulty is that, due to particle conservation, space–time correlations in the field of red particles decay slowly. This places the problem in a class of random walks in dynamic random environments for which scaling laws are hard to obtain.

Keywords: Random walk; Dynamic random environment; Multi-scale renormalization; Law of large numbers

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1. Introduction and background

1.1. Model and main theorem

We consider a *green* particle that performs a continuous-time random walk on $\mathbb{Z}^d, d \geq 1$, under the influence of a field of *red* particles which themselves perform independent continuous-time simple random walks jumping at rate 1, constituting a dynamic random environment. The latter is denoted by

$$N = (N(t))_{t\geq 0} \quad \text{with } N(t) = \{N(x,t)\colon x \in \mathbb{Z}^d\}, \tag{1.1}$$

where $N(x,t) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the number of red particles at site x at time t. As initial state we take $N(0) = \{N(x,0)\colon x \in \mathbb{Z}^d\}$ to be i.i.d. Poisson random variables with mean μ. As is well known, this makes N invariant under translations in space and time.

Also the green particle jumps at rate 1, however, our assumption is that the jump is drawn from two different random walk transition kernels p' and p'' on $\mathbb{Z}^d$ depending on whether the space–time point of the jump is occupied by a red particle or not. We assume that p' and p'' have finite range, and write

$$v' = \sum_{x\in\mathbb{Z}^d} xp'(0,x), \qquad v'' = \sum_{x\in\mathbb{Z}^d} xp''(0,x), \tag{1.2}$$

to denote their mean. We write

$$\mathcal{G} = (\mathcal{G}(t))_{t\geq 0} \tag{1.3}$$

to denote the path of the green particle with $\mathcal{G}(0) = 0$, and write P^μ to denote the joint law of N and $\mathcal{G}$. Our main result is the following asymptotic weak law of large numbers ($\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$).

Theorem 1.1. *For every $\varepsilon > 0$,*

$$\lim_{\mu\to\infty} \limsup_{t\to\infty} P^\mu\{\|t^{-1}\mathcal{G}(t) - v'\| > \varepsilon\} = 0. \tag{1.4}$$

1.2. Discussion

The result in Theorem 1.1 is far from surprising. As $\mu \to \infty$, at any given time the fraction of *sites* occupied by red particles tends to 1. Therefore we may expect that the fraction of *time* the green particle sees a red particle tends to 1 also. Consequently, we may expect the green particle to almost satisfy a weak law of large numbers corresponding to the transition kernel p', as if it were seeing a red particle always. Despite this simple intuition, the result in Theorem 1.1 seems non-trivial to prove. The proof in the present note relies on techniques developed in [11] to deal with spread-of-infection models.

The key problem is to show that for large μ the green particle is unlikely to spend an appreciable amount of time in the rare space–time holes of the field of red particles. To see why this is non-trivial, consider the case $d = 1$ with two nearest-neighbor transition kernels p' and p'' of the form

$$p'(0,1) = p = p''(0,-1), \qquad p'(0,-1) = 1-p = p''(0,1), \quad p \in \left(\tfrac{1}{2}, 1\right), \tag{1.5}$$