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On the approximation by three consecutive continued fraction convergents

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Abstract

Denote by $p_n/q_n, n = 1, 2, 3, \ldots,$ the sequence of continued fraction convergents of the real irrational number x. Define the sequence of approximation coefficients by $\theta_n := q_n |q_n x - p_n|, n = 1, 2, 3, \ldots$. A laborious way of determining the mean value of the sequence $|\theta_{n+1} - \theta_{n-1}|, n = 2, 3, \ldots,$ is simplified. The method involved also serves for showing that for almost all x the pattern $\theta_{n-1} < \theta_n < \theta_{n+1}$ occurs with the same asymptotic frequency as the pattern $\theta_{n+1} < \theta_n < \theta_{n-1}$, namely $0.12109\cdots$. All the four other patterns have the same asymptotic frequency $0.18945\cdots$. The constants are explicitly given.

Keywords: Continued fractions; Metric theory

1. Introduction

Let $x \in \Omega := [0, 1] \setminus \mathbb{Q}$ have the continued fraction expansion

$$x = [0; a_1, a_2, \ldots]$$

and let $p_n/q_n,\ n = 1, 2, 3, \ldots,$ be the corresponding sequence of convergents. The sequence θ_n, $n = 1, 2, 3, \ldots,$ of approximation coefficients of x is defined by

$$\theta_n(x) := q_n^2 \left| x - \frac{p_n}{q_n} \right|, \quad n = 1, 2, 3, \ldots.$$

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In our notation we will omit the dependence of θ_n on x. An important aspect of the approximation by continued fractions is the fact that

$$0 < \theta_n < 1, \quad n = 1, 2, 3, \ldots.$$

The sequence has been studied extensively; see for instance [2, Chapter 5].

Let the operator $T : \Omega \to \Omega$ be defined by

$$Tx := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Since for $x = [0; a_1, a_2, a_3, \ldots]$ we have $Tx = [0; a_2, a_3, \ldots]$, T is called the *one-sided shift operator* connected with the continued fraction, also known as the *Gauss operator*.

Now let $\bar{\Omega} := \Omega \times [0, 1]$ and $x = [0; a_1, a_2, \ldots]$. The natural extension of T is the *two-sided shift operator* $\mathfrak{T} : \bar{\Omega} \to \bar{\Omega}$, defined by

$$\mathfrak{T}(x, y) := \left(Tx, \frac{1}{y + a_1} \right) = \left(\frac{1}{x} - a_1, \frac{1}{y + a_1} \right).$$

In particular, $\mathfrak{T}$ is measure-preserving with regard to the measure m with density function μ, where

$$\mu(x, y) := \frac{1}{\log 2} \cdot \frac{1}{(1 + xy)^2}. \tag{1}$$

We define

$$(t_n, v_n) := \mathfrak{T}^n(x, 0), \quad n \geq 1,$$

that is

$$t_n = [0; a_{n+1}, a_{n+2}, \ldots]$$

and

$$v_n = [0; a_n, a_{n-1}, \ldots, a_1] = \frac{q_{n-1}}{q_n}.$$

Beside the well-known relations (see e.g. [5, p. 25])

$$\theta_{n-1} = \frac{v_n}{1 + t_n v_n} \tag{2}$$

and

$$\theta_n = \frac{t_n}{1 + t_n v_n}, \tag{3}$$

we will need the relation

$$\theta_{n+1} = \frac{(1 - a_{n+1} t_n)(a_{n+1} + v_n)}{1 + t_n v_n}; \tag{4}$$

see [2, Proposition 5.3.6].

In this section we will prove various arithmetical properties of $\mathfrak{T}$, from which we will later deduce some metrical results on triples of three consecutive approximation coefficients.