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Spectral analysis for the Gauss problem on continued fractions

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Abstract

We present a new derivation of the formula appearing in Babenko (1978) and Mayer and Roepstorff (1987) that gives the probability distribution of τ^{-n} in terms of the eigenvalues of a symmetric operator. Here τ is the well-known Gauss-map.

Keywords: Continued fraction; The Gauss problem; Perron–Frobenius operator

1. Introduction

The theory of continued fractions is built on the Gauss-transformation τ, defined as

$$\tau(x) = \begin{cases} \{1/x\} & \text{if } x \in I = [0, 1], \ x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\{\cdot\}$ denotes the fractional part.

The probability measure $\gamma(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{x+1}$, $A \in \mathcal{B}_I$ = the collection of Borel sets in I, known as the Gauss measure, is τ-invariant, i.e., $\gamma\tau^{-1} = \gamma$, to mean $\gamma(\tau^{-1}(A)) = \gamma(A)$ for any $A \in \mathcal{B}_I$. Denote by $[\cdot]$ the integer part. Define $a_1(x) = \left[\frac{1}{x}\right]$, $x \in (0, 1]$, $a_1(0) = \infty$, and $a_n(x) = a_1(\tau^{n-1}(x))$, $x \in I, n \in \mathbb{N} = \{1, 2, \ldots\}$, with $\tau^0 = \text{id}$. Then

$$x = \frac{1}{[1/x] + \{1/x\}} = \frac{1}{a_1(x) + \tau(x)}$$

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for any $x \in I$. Hence

$$\tau(x) = \frac{1}{a_1(\tau(x)) + \tau(\tau(x))} = \frac{1}{a_2(x) + \tau^2(x)}$$

and thus

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \tau^2(x)}}.$$

Continuing in this manner we obtain

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots + \cfrac{1}{a_n(x) + \tau^n(x)}}}}$$

for any $x \in I$ and $n \in \mathbb{N}$.

This is the continued fraction expansion of $x \in I$. When x is a rational number there exists an integer $n \in \mathbb{N}$ such that $\tau^m(x) = 0$ for any $m \geq n$. In this case the continued fraction of x contains a finite number of finite incomplete quotients $a_1(x), a_2(x), \ldots$. We shall use the notation $x = [a_1(x), a_2(x), \ldots]$ for the number x with incomplete quotients $a_1(x), a_2(x), \ldots$.

Clearly, by the equation $\gamma\tau^{-1} = \gamma$, the sequence $(a_n)_{n\in\mathbb{N}}$ is a strictly stationary one when we place the Gauss measure γ on $\mathcal{B}_I$. It is possible to give a representation of the incomplete quotients as a doubly infinite sequence as follows. For $(x, y) \in I \times I$ put $\bar{a}_n(x, y) = a_n(x), \bar{a}_0(x, y) = a_1(y), \bar{a}_{-n}(x, y) = a_{n+1}(y), n \in \mathbb{N}$. Then the doubly infinite sequence $(\bar{a}_l)_{l\in\mathbb{Z}}, \mathbb{Z} = (\ldots, -2, -1, 0, 1, 2, \ldots)$, is a doubly infinite version of $(a_n)_{n\in\mathbb{N}}$ on the probability space $(I \times I, \mathcal{B}_{I\times I}, \bar{\gamma})$ where $\bar{\gamma}$ is the probability measure with density $\frac{1}{(xy+1)^2 \log 2}$, $x, y \in I$. Remark that, in fact, we are dealing with the natural extension of the dynamical system underlying the one-dimensional system of the regular continued fraction expansion. See [3, Subsection 1.3.1 and p. 31]. We have

$$\bar{\gamma}([0, u] \times I | \bar{a}_0, \bar{a}_{-1}, \ldots) = \frac{(a+1)u}{au+1}, \quad u \in I, \bar{\gamma}\text{-a.e.}, \tag{1}$$

where $a = [\bar{a}_0, \bar{a}_{-1}, \ldots]$. See [3, Theorem 1.3.5].

In the next section we introduce the Perron–Frobenius operator of τ under $\bar{\gamma}$. In Sections 5 and 6 we will show that a spectral decomposition of it does exist.

2. The Perron–Frobenius operator of τ

The Perron–Frobenius operator of τ under a probability measure μ on $\mathcal{B}_I$ such that $\mu\big(\tau^{-1}(A)\big) = 0$ whenever $\mu(A) = 0$ is defined as the bounded linear operator P_μ on $L^1_\mu(I)$ which takes $f \in L^1_\mu(I)$ into $P_\mu f \in L^1_\mu(I)$ with

$$\int_A P_\mu \phi \, d\mu = \int_{\tau^{-1}(A)} \phi \, d\mu$$