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indagationes mathematicae

Indagationes Mathematicae 23 (2012) 269–299

www.elsevier.com/locate/indag

The prime-pair conjectures of Hardy and Littlewood

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Received 25 October 2011; accepted 9 December 2011

Communicated by F. Beukers

Abstract

By (extended) Wiener–Ikehara theory, the prime-pair conjectures are equivalent to simple pole-type boundary behavior of corresponding Dirichlet series. Under a weak Riemann-type hypothesis, the boundary behavior of weighted sums of the Dirichlet series can be expressed in terms of the behavior of certain double sums $\Sigma^*_{2k}(s)$. The latter involve the complex zeros of $\zeta(s)$ and depend in an essential way on their differences. Extended prime-pair conjectures are true if and only if the sums $\Sigma^*_{2k}(s)$ have good boundary behavior. Equivalently, a more general sum $\Sigma^*_{\omega}(s)$ (with real $\omega > 0$) should have a boundary function (or distribution) that is well-behaved, apart from a pole $R(\omega)/(s - 1/2)$ with residue $R(\omega)$ of period 2. [$R(\omega)$ could be determined for $\omega \le 2$.]

Keywords: Prime-pair conjecture; Wiener–Ikehara theorem; Zeta zeros

1. Introduction

Most mathematicians believe that there are infinitely many *prime twins* $(p, p+2)$, although this has not been proved. In fact, there is strong numerical support for the prime-pair conjectures ("PPC's") B and D of Hardy and Littlewood [12]. Conjecture B asserts that the number $\pi_{2r}(x)$ of prime pairs $(p, p+2r)$ with $p \le x$ satisfies the asymptotic relation

$$\pi_{2r}(x) \sim 2C_{2r}\,\mathrm{li}_2(x) = 2C_{2r}\int_2^x \frac{dt}{\log^2 t} \sim 2C_{2r}\frac{x}{\log^2 x} \tag{1.1}$$

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doi:10.1016/j.indag.2011.12.001

Table 1
Counting prime pairs $(p, jp \pm 2r)$ with $p \le x$.

prprs $\setminus x$	10^3	10^4	10^5	10^6	10^7	10^8	C_{2jr}/C_2
$(p, p+2)$	35	205	1224	8 169	58 980	440 312	1
$(p, p+4)$	41	203	1216	8 144	58 622	440 258	1
$(p, p+6)$	74	411	2447	16 386	117 207	879 908	2
$(p, p+8)$	38	208	1260	8 242	58 595	439 908	1
$(p, p+10)$	51	270	1624	10 934	78 211	586 811	4/3
$(p, p+12)$	70	404	2421	16 378	117 486	880 196	2
$(p, p+14)$	48	245	1488	9 878	70 463	528 095	6/5
$(p, p+16)$	39	200	1233	8 210	58 606	441 055	1
$(p, p+30)$	99	536	3329	21 990	156 517	1 173 934	8/3
$(p, p+210)$	107	641	3928	26 178	187 731	1 409 150	16/5
$(p, 3p+2)$	64	352		15 136		828 477	2
$(p, 3p-2)$	64	362		15 007		826 250	2
$(p, 9p+2)$	57	342		14 003		780 760	2
$(p, 9p-2)$	52	310		13 928		781 433	2
$L_2(x)$:	46	214	1249	8 248	58 754	440 368	

as $x \to \infty$. Here

$$C_2 = \prod_{p>2 \text{ prime}} \left\{1 - \frac{1}{(p-1)^2}\right\} \approx 0.6601618, \tag{1.2}$$

and

$$C_{2r} = C_2 \prod_{q>2 \text{ prime};\, q|r} \frac{q-1}{q-2}. \tag{1.3}$$

Thus, for example, $C_4 = C_8 = C_2$, $C_6 = 2C_2$, $C_{10} = (4/3)C_2$. We mention the curious fact that the prime-pair constants C_{2r} have *mean value* 1. Bombieri and Davenport [4], and later, Friedlander and Goldston [8], gave precise estimates; Tenenbaum [26] recently found a simple proof.

On the Internet one finds counts of twin primes for p up to 10^{16} by Nicely [22]. In Amsterdam, prime pairs $(p, p+2r)$ have been counted by Fokko van de Bult [29] and Herman te Riele [21]; the latter has also counted certain prime pairs $(p, jp \pm 2r)$ [23]. Table 1 shows a very small part of their work; the bottom line shows (rounded) values $L_2(x)$ of the comparison function $2C_2\mathrm{li}_2(x)$. Tables support the strong conjecture that for every r and $\varepsilon > 0$,

$$\pi_{2r}(x) - 2C_{2r}\mathrm{li}_2(x) = \mathcal{O}\left\{x^{(1/2)+\varepsilon}\right\}. \tag{1.4}$$

[The corresponding conjecture for $\pi(x)$, the number of primes $p \le x$, is equivalent to Riemann's Hypothesis (RH).]

Among other things, the Hardy–Littlewood Conjecture D deals with prime pairs $(p, jp \pm 2r)$, where j is *prime* to $2r$. The corresponding counting functions $\pi_{j,\pm 2r}(x)$ for pairs with $p \le x$ should be roughly comparable to $2C_{2jr}\,\mathrm{li}_2(x)$, but see (1.8). Conjectures by later authors involved still more general prime pairs; we mention Schinzel and Sierpinski [25], Bateman and Horn [2,3] and Schinzel [24]; cf. also the survey by Hindry and Rivoal [15].

It is a classical result of Brun [5], obtained by applying what is now called Brun's sieve, that $\pi_2(x) = \mathcal{O}(x/\log^2 x)$. Using more advanced sieves, Jie Wu [33] has shown that $\pi_2(x) < 6.8\, C_2\, x/\log^2 x$ for all sufficiently large x. There are related results for prime pairs $(p, jp \pm 2r)$.