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Real projective structures on a real curve

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Abstract

Given a compact connected Riemann surface X equipped with an antiholomorphic involution τ, we consider the projective structures on X satisfying a compatibility condition with respect to τ. For a projective structure P on X, there are holomorphic connections and holomorphic differential operators on X that are constructed using P. When the projective structure P is compatible with τ, the relationships between τ and the holomorphic connections, or the differential operators, associated to P are investigated. The moduli space of projective structures on a compact oriented C^∞ surface of genus $g \geq 2$ has a natural holomorphic symplectic structure. It is known that this holomorphic symplectic manifold is isomorphic to the holomorphic symplectic manifold defined by the total space of the holomorphic cotangent bundle of the Teichmüller space $\mathcal{T}_g$ equipped with the Liouville symplectic form. We show that there is an isomorphism between these two holomorphic symplectic manifolds that is compatible with τ.

Keywords: Projective structure; Real curve; Connection; Differential operator

1. Introduction

A projective structure on a compact Riemann surface X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are Möbius transformations.

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Projective structures have other equivalent formulations using projective connections, differential operators etc.

Assume that there is an antiholomorphic involution

$$\tau : X \longrightarrow X.$$

Just as compact Riemann surfaces are same as smooth projective curves defined over $\mathbb{C}$, pairs of the form (X, τ) are same as geometrically irreducible smooth projective curves over $\mathbb{R}$. We consider projective structures on X compatible with τ; a projective structure P on X is compatible with τ if τ takes any holomorphic coordinate function associated to P to the conjugate of another holomorphic coordinate function associated to P. So a projective structure on X compatible with τ can be called a *real* projective structure.

This "reality" of projective structures becomes more clear if we consider the equivalent formulations using projective connections or differential operators. Associated to each projective structure is a holomorphic (equivalently, algebraic) projective connection. A projective structure is real if and only if the corresponding projective connection is defined over $\mathbb{R}$. Also, associated to each projective structure is a holomorphic (equivalently, algebraic) differential operator of order three from TX to $((TX)^*)^{\otimes 2}$, where TX is the holomorphic tangent bundle. A projective structure is real if and only if the corresponding differential operator is defined over $\mathbb{R}$. We investigate these interrelations.

To define a projective structure on a C^∞ oriented surface, we do not need to fix a complex structure on the surface. On the contrary, given a projective structure, there is a underlying complex structure on the surface.

Let Y be a compact connected oriented C^∞ surface. Let $\mathrm{Diffeo}_0(Y)$ denote the group of diffeomorphisms of Y homotopic to the identity map. This group acts on the space of all projective structures on Y compatible with the orientation of Y. The corresponding quotient space will be denoted by $\mathcal{P}_0(Y)$, which is a complex manifold equipped with a natural holomorphic symplectic form. The Teichmüller space $\mathcal{T}(Y)$ for Y is the quotient by $\mathrm{Diffeo}_0(Y)$ of the space of all complex structures on Y compatible with its orientation. There is a natural projection of

$$\varphi : \mathcal{P}_0(Y) \longrightarrow \mathcal{T}(Y)$$

making $\mathcal{P}_0(Y)$ a torsor for the holomorphic cotangent bundle $\Omega^1_{\mathcal{T}(Y)} \longrightarrow \mathcal{T}(Y)$. Let

$$\tau : Y \longrightarrow Y$$

be an orientation reversing diffeomorphism of order two. Let τ_P (respectively, τ_T) be the involution of $\mathcal{P}_0(Y)$ (respectively, $\mathcal{T}(Y)$) constructed using τ. We prove that there is a holomorphic section of the projection φ that

- intertwines τ_P and τ_T, and
- the corresponding biholomorphism of $\mathcal{P}_0(Y)$ with $\Omega^1_{\mathcal{T}(Y)}$ takes the natural symplectic form on $\mathcal{P}_0(Y)$ to the Liouville symplectic form on $\Omega^1_{\mathcal{T}(Y)}$.

In a work with Huisman, [4], the representations associated to stable real vector bundles are introduced. The PGL_2 analog of these representations arise in real projective structures.

2. Real projective structures

Let X be a compact connected Riemann surface. Let

$$J : T^{\mathbb{R}}X \longrightarrow T^{\mathbb{R}}X$$