Available online at www.sciencedirect.com
SciVerse ScienceDirect
Indagationes Mathematicae 23 (2012) 361–376
indagationes mathematicae
www.elsevier.com/locate/indag

Some remarks on the geometry of Lorentz spaces $\Lambda_{1,w}$

Paweł Foralewski

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

Received 14 November 2011; received in revised form 16 December 2011; accepted 1 February 2012

Communicated by B. de Pagter

Abstract

Some geometric properties of classical Lorentz spaces $\Lambda_{1,w}$ are considered. First criteria for the Kadec–Klee property with respect to the local convergence in measure for Lorentz spaces $\Lambda_{1,w}$ are given. In order to prove these criteria it was necessary to find first weaker sufficient conditions for the almost everywhere convergence of a sequence of rearrangements (x_n^*) to a rearrangement element x^*. Next criteria for non-squareness as well as for extreme points of the unit ball of the spaces are established. The last result is a generalization of the result presented in Carothers et al. (1992) [5].

Keywords: Lorentz function space; Order continuity; Separability; Kadec–Klee property; Strict monotonicity; Lower locally uniform monotonicity; Upper locally uniform monotonicity; Non-squareness; Locally uniform non-squareness; Uniform non-squareness; Extreme point; Rotund point; Strong U-point; Rotundity

1. Preliminaries

Lorentz spaces introduced by Lorentz in 1950 (see [37,38]) play an important role in the theory of Banach spaces, in particular they are key objects in the interpolation theory of linear operators. Since their introduction Lorentz spaces have been objects of extensive investigations, results of which are contained among others in the papers [19,20,1,4–6,22,7,32,29,11,30] and the monographs [2,34–36].

Let $L^0 = L^0([0,\gamma), \Sigma, m)$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0,\gamma)$, where $\gamma \le \infty$. For any $x, y \in L^0$, we write $x \le y$, if $x(t) \le y(t)$ almost everywhere with respect to the Lebesgue measure m on $[0,\gamma)$.

E-mail address: katon@amu.edu.pl.

0019-3577/$ - see front matter
doi:10.1016/j.indag.2012.02.002

Given any $x \in L^0$ we define its distribution function $\mu_x : [0, +\infty) \to [0, \gamma]$ by

$$\mu_x(\lambda) = m(\{t \in [0, \gamma) : |x(t)| > \lambda\})$$

(see [2,34,36]) and the non-increasing rearrangement $x^* : [0, \gamma) \to [0, \infty]$ of x as

$$x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\}$$

(under the convention $\inf \emptyset = \infty$). We say that two functions $x, y \in L^0$ are equimeasurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. Then we obviously have $x^* = y^*$.

A Banach space $E = (E, \leq, \|\cdot\|)$, where $E \subset L^0$, is said to be a Köthe space if the following conditions are satisfied:

(i) if $x \in E$, $y \in L^0$ and $|y| \leq |x|$, then $y \in E$ and $\|y\| \leq \|x\|$,
(ii) there exists a function x in E that is strictly positive on the whole $[0, \gamma)$.

Recall that the Köthe space E is called symmetric if E is rearrangement invariant in the sense that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$ (see [7]). For basic properties of symmetric spaces we refer to [2,34,36].

By w we denote a nonnegative and locally integrable function on $[0, \gamma)$ (not identically 0) called a weight function and define on L^0 the functional

$$\|x\| = \int_0^\gamma x^*(t)w(t)dt$$

with values in $[0, +\infty]$. It is well known that the functional $\|\cdot\|$ satisfies the triangle inequality and so it is a function norm (recall that function norm admits values $+\infty$) if and only if the weight function w is a non-increasing on the interval $[0, \gamma)$ (see [38, Theorem 1]).

Assuming in the following that w is non-increasing function, we define the Lorentz space $\Lambda_{1,w}$ by the formula

$$\Lambda_{1,w} = \{x \in L^0 : \|x\| < \infty\}$$

(see [38]). It is well known that $\Lambda_{1,w}$ is a Banach symmetric space (see [38, page 413]).

The remark presented below will be used in the third part of the paper.

Remark 1. Let $x, y \in \Lambda_{1,w}$ and $t \in (0, \gamma)$ be such that $(\frac{x+y}{2})^*(t) > \lim_{s\to\infty}(\frac{x+y}{2})^*(s)$. Then, by Krein et al. ([34], property 7°, page 64), there is a set $e_t = e_t(\frac{x+y}{2})$ such that $m(e_t) = t$ and

$$\int_0^t \left(\frac{x+y}{2}\right)^*(s)ds = \int_{e_t} \left|\frac{x+y}{2}\right|(s)ds.$$

Defining $t(x) = m(\operatorname{supp} x \cap e_t)$ and $t(y) = m(\operatorname{supp} y \cap e_t)$, by convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}\int_0^t \left(\frac{x+y}{2}\right)^*(s)w(s)ds &= \left\|\left(\frac{x+y}{2}\right)\chi_{e_t}\right\| \leq \frac{1}{2}\|x\chi_{e_t}\| + \frac{1}{2}\|y\chi_{e_t}\| \\ &= \frac{1}{2}\int_0^{t(x)} (x\chi_{e_t})^*(s)w(s)ds + \frac{1}{2}\int_0^{t(y)} (y\chi_{e_t})^*(s)w(s)ds.\end{aligned}$$

Denoting $A_t = [0, \gamma)\backslash e_t$, $a(x) = m(\operatorname{supp} x \cap A_t)$, $a(y) = m(\operatorname{supp} y \cap A_t)$ and applying convexity of the norm $\|\cdot\|_t$, defined by the formula

$$\|x\|_t = \int_0^\gamma x^*(s)w(t+s)ds$$