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Constructing the minimal period of homomorphisms into $\mathbf{R}^n$

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Abstract

Following on from the work of Bridges and Hendtlass (2010) [5], we provide geometric conditions under which the minimal period of a continuous periodic homomorphism from $\mathbf{R}$ onto a nontrivial metric abelian group contained in $\mathbf{R}^n$ can be constructed within the framework of Bishop's constructive mathematics.

Keywords: Constructive analysis; Periodic homomorphism; Minimal period

In this paper we consider the classically vacuous problem of finding the minimal period of a continuous periodic homomorphism from $\mathbf{R}$ onto a nontrivial complete abelian group G within the framework of Bishop's constructive mathematics (**BISH**).[1] Let θ be a homomorphism with domain $\mathbf{R}$. We say that θ is *periodic* if there exists $\tau > 0$, a *period of* θ, such that $\theta(\tau) = \mathbf{0}_G$, where $\mathbf{0}_G$ denotes the identity element of G; if also $\theta(t)$ is apart from $\mathbf{0}_G$ – denoted as $\theta(t) \neq \mathbf{0}_G$ – for each $t \in (0, \tau)$, then we call τ the *minimal period of* θ. We denote the minimal period of a homomorphism by $\tau_{\min}$. In [5] it was shown that given a continuous periodic homomorphism from $\mathbf{R}$ onto a nontrivial compact metric abelian group, we cannot, in general, find the minimal period, even when we adopt a definition of the minimal period constructively weaker than that used here. In this paper we seek conditions under which the minimal period does exist; we prove that if θ is continuously differentiable and periodic, and G satisfies a simple geometric condition, then we can construct the minimal period of θ.

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[1] That is, mathematics with intuitionistic logic and an appropriate set-theoretic foundation such as those in [1,4,9,10]. For more on **BISH** and other varieties of constructive mathematics, see [3,6,7].

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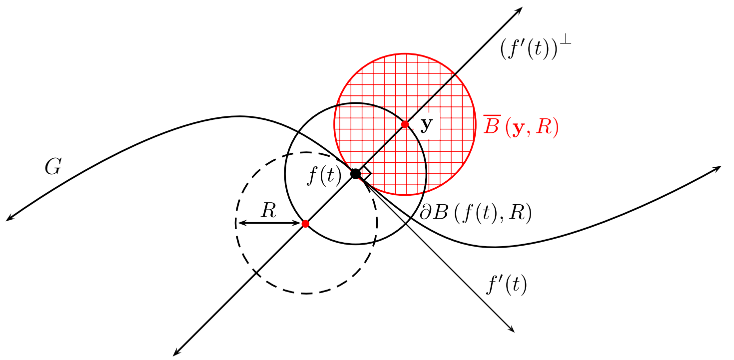

Fig. 1. The encircled tangent ball condition in two dimensions.

A *metric abelian group*[2] is an abelian group G equipped with a metric such that the mapping $(x, y) \rightsquigarrow y - x$ is pointwise continuous at $(\mathbf{0}_G, \mathbf{0}_G) \in G \times G$, and uniformly continuous on compact[3] subsets of $G \times G$. The mappings $x \rightsquigarrow -x$ and $(x, y) \rightsquigarrow x + y$ are then pointwise continuous throughout their domains, and uniformly continuous on compact subsets of their domains. A metric abelian group G is said to be *nontrivial* if the metric complement of $\{\mathbf{0}_G\}$, in G, is inhabited.

A homomorphism θ of the additive group $\mathbf{R}$ into a metric abelian group G is *continuous* if it is uniformly continuous on each compact (or, equivalently, on each bounded) subset of $\mathbf{R}$. In particular, if θ is periodic, then it is uniformly continuous on $\mathbf{R}$.

Let G be the image of $\mathbf{R}$ under a differentiable map f into $\mathbf{R}^n$. Then G is said to satisfy the *encircled tangent ball condition*[4] if for each $t \in \mathbf{R}$ there exists $R > 0$ such that for all $\mathbf{y} \in (f'(t))^{\perp} \cap \partial B(f(t), R)$ we have

$$\overline{B}(\mathbf{y}, R) \cap G = \{f(t)\},$$

where $B(x, r)$ and $\overline{B}(x, r)$ represent the open and closed balls, respectively, centered on x with radius r, and ∂S represents the boundary of S. We say that R is a *buffer radius of G at $\theta(t)$*. If there exists $R > 0$ such that R is a buffer radius for G at x for each x in G, then we say that G satisfies the *uniform encircled tangent ball condition*. The encircled tangent ball condition provides a lower bound for the radius of curvature of f at each point; more importantly it also bounds how close the curve can get to itself without becoming periodic. Fig. 1 illustrates the encircled tangent ball condition in two dimensions.

At first blush, the encircled tangent ball condition seems to be unnecessarily complicated, and somewhat contrived. A more natural, though still quite complicated, condition to demand is that f be *locally bijective*:

> For each $s \in \mathbf{R}$ there exist $\varepsilon > 0$ and $t, t' \in \mathbf{R}$ such that $s \in (t, t')$ and f is a bijection between (t, t') and $B(f(s), \varepsilon)$.

[2] We use the standard additive notation for all abelian groups.

[3] A subset S of a metric space is totally bounded if it can be covered by arbitrarily small balls centered on points in S, and is compact if it is both complete and totally bounded. In the presence of Brouwer's fan theorem this definition of compactness is equivalent to open cover compactness, and with weak König's lemma it is equivalent to sequential compactness (see [6]). Any closed ball in $\mathbf{R}^N$ is compact.

[4] This is a generalization of the twin tangent ball condition introduced in [8].