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Smoothness and uniqueness in ridge function representation

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Abstract

In this note we consider problems of uniqueness, smoothness and representation of linear combinations of a finite number of ridge functions with fixed directions.
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1. Introduction

A *ridge function*, in its simplest format, is a multivariate function of the form

$$f(\mathbf{a}\cdot\mathbf{x}),$$

defined for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is a fixed non-zero vector, called a *direction*, $\mathbf{a}\cdot\mathbf{x} = \sum_{j=1}^n a_j x_j$ is the usual inner product, and f is a real-valued function defined on $\mathbb{R}$. Note that

$$f(\mathbf{a}\cdot\mathbf{x})$$

is constant on the hyperplanes $\{\mathbf{x} : \mathbf{a}\cdot\mathbf{x} = c\}$. Ridge functions are relatively simple multivariate functions. Ridge functions (formerly known as *plane waves*) were so-named in 1975 by Logan and Shepp [11]. They appear in various areas and under numerous guises.

In this note we consider problems of uniqueness, smoothness and representation of linear combinations of a finite number of ridge functions. That is, assume we are given a function F of

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the form

$$F(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{a}^i \cdot \mathbf{x}), \tag{1.1}$$

where m is finite, and the $\mathbf{a}^i$ are pairwise linearly independent vectors in $\mathbb{R}^n$. We ask and answer the following questions. If F is of a certain smoothness class, what can we say about the smoothness of the f_i? How many different ways can we write F as a linear combination of a finite number of ridge functions, i.e., to what extent is a representation of F in the form (1.1) unique? And, finally, which other ridge functions $f(\mathbf{a} \cdot \mathbf{x})$ can be written in the form (1.1) with $\mathbf{a} \neq \alpha \mathbf{a}^i$, for any $\alpha \in \mathbb{R}$ and $i = 1, \ldots, m$?

In Section 4 we generalize the main results of this paper to finite linear combinations of functions of the form

$$f(A\mathbf{x})$$

where A is a fixed $d \times n$ matrix, $1 \leq d < n$, and f is a real-valued function defined on $\mathbb{R}^d$. For $d = 1$, this reduces to a ridge function.

2. Smoothness

Let $C^k(\mathbb{R}^n)$, $k \in \mathbb{Z}_+$, denote the usual set of real-valued functions with all derivatives of order up to and including k being continuous. Assume $F \in C^k(\mathbb{R}^n)$ is of the form (1.1). What does this imply, if anything, about the smoothness of the f_i? In the case $m = 1$ there is nothing to prove. That is, if

$$F(\mathbf{x}) = f_1(\mathbf{a}^1 \cdot \mathbf{x})$$

is in $C^k(\mathbb{R}^n)$ for some $\mathbf{a}^1 \neq \mathbf{0}$, then obviously $f_1 \in C^k(\mathbb{R})$. This same result holds when $m = 2$. As the $\mathbf{a}^1$ and $\mathbf{a}^2$ are linearly independent, there exists a vector $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{a}^1 \cdot \mathbf{c} = 0$ and $\mathbf{a}^2 \cdot \mathbf{c} = 1$. Thus

$$F(t\mathbf{c}) = f_1(\mathbf{a}^1 \cdot t\mathbf{c}) + f_2(\mathbf{a}^2 \cdot t\mathbf{c}) = f_1(0) + f_2(t).$$

As $F(t\mathbf{c})$ is in $C^k(\mathbb{R})$, as a function of t, so is f_2. The same result holds for f_1.

However this result is no longer valid when $m \geq 3$, without some assumptions on the f_i. To see this, let us recall that the Cauchy Functional Equation

$$g(x + y) = g(x) + g(y) \tag{2.1}$$

has, as proved by Hamel [8] in 1905, very badly behaved solutions; see e.g., Aczél [1] for a discussion of the solutions of this equation. As such, setting $f_1 = f_2 = -f_3 = g$, we have very badly behaved (and certainly not in $C^k(\mathbb{R})$) f_i, $i = 1, 2, 3$, that satisfy

$$0 = f_1(x_1) + f_2(x_2) + f_3(x_1 + x_2)$$

for all $(x_1, x_2) \in \mathbb{R}^2$. That is, the very smooth function on the left-side of this equation is a sum of three unruly ridge functions. As shall shortly become evident, this Cauchy Functional Equation is critical in the analysis of our problem for all $m \geq 3$.

It was proved by Buhmann and Pinkus [2] that if $F \in C^k(\mathbb{R}^n)$, and if $f_i \in L^1_{\text{loc}}(\mathbb{R})$ for each i, then $f_i \in C^k(\mathbb{R})$ for each i, if $k \geq m - 1$. The method of proof therein used smoothing and generalized functions. In this note we remove the restriction $k \geq m - 1$, have different

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