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Review

Uniform asymptotic methods for integrals

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Abstract

We give an overview of basic methods that can be used for obtaining asymptotic expansions of integrals: Watson's lemma, Laplace's method, the saddle point method, and the method of stationary phase. Certain developments in the field of asymptotic analysis will be compared with De Bruijn's book *Asymptotic Methods in Analysis*. The classical methods can be modified for obtaining expansions that hold uniformly with respect to additional parameters. We give an overview of examples in which special functions, such as the complementary error function, Airy functions, and Bessel functions, are used as approximations in uniform asymptotic expansions.

Keywords: Asymptotic analysis; Method of stationary phase; Saddle point method; Uniform asymptotic expansions; Special functions

Contents

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1. Introduction

Large parameter problems occur in all branches of pure and applied mathematics, in physics and engineering, in statistics and probability theory. On many occasions the problems are presented in the form of integrals or differential equations, or both, but we also encounter finite sums, infinite series, difference equations, and implicit algebraic equations.

Asymptotic methods for handling these problems have a long history with many prominent contributors. In 1863, Riemann used the method of steepest descent for hypergeometric functions, and in 1909 Debye [6] used this method to obtain approximations for Bessel functions. In other unpublished notes, Riemann also gave the first steps for approximating the zeta function and in 1932 Siegel used this method to derive the Riemann–Siegel formula for the Riemann zeta function. The method of stationary phase was essential in Kelvin's work to describe the wave pattern behind a moving ship.

In the period 1940–1950, a systematic study of asymptotic methods started in the Netherlands, driven by J.G. van der Corput, who considered these methods important when studying problems of number theory in his earlier years in Groningen. Van der Corput was one of the founders of the Mathematisch Centrum in Amsterdam, and in 1946 he became the first director. He organized working groups and colloquia on asymptotic methods and he had much influence on workers in this area. During 1950–1960 he wrote many papers on asymptotic methods and during his years in the USA he published lecture notes and technical reports.

Nowadays his work on asymptotic analysis for integrals is still of interest, in particular his work on the method of stationary phase. He introduced for this topic the so-called neutralizer in order to handle integrals with several points that contribute to the asymptotic behavior of the complete integral. In later sections we give more details.

Although Van der Corput wanted[2] to publish a general compendium to asymptotic methods, the MC lecture notes and his many published papers, notes and reports were never combined into

[2] He concluded a Rouse Ball lecture at Cambridge (England) in 1948 with a request [38] to all workers in asymptotics to send information to the MC, and a study group would include these contributions in a general survey of the whole field.