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Nicolaas Govert de Bruijn, the enchanter of friable integers

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In memoriam: Nicolaas Govert ('Dick') de Bruijn (1918–2012)

Abstract

N.G. de Bruijn carried out fundamental work on integers having only small prime factors and the Dickman–de Bruijn function that arises on computing the density of those integers. In this he used his earlier work on linear functionals and differential–difference equations. We review his relevant work and also some later improvements by others.

Keywords: N.G. de Bruijn; Friable integers; Dickman–de Bruijn function

1. Introduction

The number theoretical work of Nicolaas Govert ('Dick') de Bruijn (1918–2012) comes into two highly distinct flavours: combinatorial and analytical. In his combinatorial number theoretical work de Bruijn even did some cowork with one of the all-time greats in this area: Paul Erdős (6 joint papers!). Here we will only discuss de Bruijn's work in analytic number theory. This was done mainly in two periods: 1948–1953 and 1962–1966. In the second period de Bruijn revisited his earlier subjects. Some of the later work is joint with Jacobus Hendricus ('Jack') van Lint (1932–2004) [34,99].

In *sieve theory* (see, e.g., [46,52] or for a very brief introduction Section 6 below) one is interested in estimating in terms of elementary functions the number of integers having a prescribed factorization in a prescribed sequence. Usually one is interested in the integers in the

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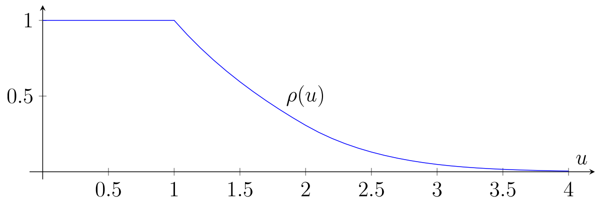

Fig. 1. The Dickman–de Bruijn function $\rho(u)$.

sequence that are primes and if one cannot handle those, integers that have only a few distinct prime factors. De Bruijn's work in analytic number theory belongs mainly to sieve theory, but he is not restricting the number of prime factors of the integers that remain after sieving. This is a milder form of sieving and here usually quite sharp estimates can be obtained. As a basic example one can take the friable number counting function $\Psi(x, y)$. Let $P(n)$ denote the largest prime divisor of an integer $n \geqslant 2$. Put $P(1) = 1$. A number n is said to be *y-friable*[1] if $P(n) \leqslant y$. We let $S(x, y)$ denote the set of integers $1 \leqslant n \leqslant x$ such that $P(n) \leqslant y$. The cardinality of $S(x, y)$ is denoted by $\Psi(x, y)$.

In 1930, Dickman[2] [38] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{1}$$

where the *Dickman function* (today often also called *Dickman–de Bruijn function*) $\rho(u)$ is defined by

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leqslant u \leqslant 1; \\ \dfrac{1}{u}\displaystyle\int_0^1 \rho(u-t)\,\mathrm{d}t & \text{for } u > 1. \end{cases} \tag{2}$$

Note that $\rho(u) > 0$, for if not, then because of the continuity of $\rho(u)$ there is a smallest zero $u_0 > 1$ and on substituting u_0 in (2) we easily arrive at a contradiction. De Bruijn's contribution was to provide a very precise asymptotic estimate for $\rho(u)$ and in later works he provided much more precise estimates than hitherto known for $\Psi(x, y)$. As a consequence various authors call $\rho(u)$ the Dickman–de Bruijn function and some authors $\Psi(x, y)$ the Dickman–de Bruijn function. Both in depth and originality de Bruijn's work goes way beyond everything done on the subject until then. Only starting in the 1980s, when Hildebrand and Tenenbaum started to work in this area, various results of de Bruijn were greatly improved upon. Most introductions of $\Psi(x, y)$ related papers mention some de Bruijn papers. The techniques de Bruijn used in studying $\rho(u)$ are now standard in studying similar functions that arise in sieve theory; see, e.g. [46, Appendix B].

In the rest of this paper we first describe de Bruijn's work on $\rho(u)$ and then that on $\Psi(x, y)$. Before describing de Bruijn's work on $\rho(u)$ more in detail (Section 2.7), we will discuss the basic properties of $\rho(u)$ and very briefly the linear functional work of de Bruijn (Sections 2.5 and 2.6). In Section 4 de Bruijn's work on $\Psi(x, y)$ is described, after a discussion of preliminaries in Section 3. In Section 5 de Bruijn's work on $\omega(u)$, a function similar to $\rho(u)$ that arises on

[1] Some authors use y-smooth. Friable is an adjective meaning easily crumbled or broken.

[2] Karl Dickman (1861–1947) wrote his paper when he was 69 years old, after retiring from the Swedish insurance business world.