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On the factorization of Abelian groups

A.D. Sands

Department of Mathematics, Dundee University, Dundee DD1 4HN, Scotland, United Kingdom

In memoriam Nicolaas G. de Bruijn

Abstract

In this note we consider a conjecture made by Professor de Bruijn and subsequent generalizations of this conjecture. We show by using constructions of his that in a certain sense these generalizations are the best possible. A survey of these known results is presented. Finally some new results of this kind are given.

Keywords: de bruijn

1. Introduction

The study of the factorization of Abelian groups arose from the solution by Hajós [5] of a famous conjecture of Minkowski. De Bruijn wrote three papers [2,4,3] involving this topic. In this note we consider a conjecture of his in [3] concerning the factorization of finite cyclic groups. A factorization of an Abelian group G occurs when a direct sum of subsets is equal to G. If G is infinite then an infinite number of factors may arise but every factor is assumed to contain 0 and only a finite number of non-zero terms occur in any sum. The assumption that each factor contains 0 may also be assumed in the finite case since if $G = A_1 + \cdots + A_k$ is a factorization so also is $G = (A_1 - a_1) + \cdots + (A_k - a_k)$. A subset A of G is said to be periodic if there exists a non-zero element b such that $A + b = A$. The set of periods together with 0 forms a subgroup H of G. Any non-zero subgroup contained in H is called a group of periods of A. For each such subgroup K there is a subset D of A such that $A = K + D$. We shall use the notation given by de Bruijn that a factorization in which no factor is periodic is called a bad factorization.

Throughout the note the word group will be used to denote additive Abelian group. The cyclic group of order n is denoted by $Z(n)$. $|A|$ denotes the order of a finite set A. For each prime p, G_p

E-mail addresses: adsands@maths.dundee.ac.uk, ad.sands@talktalk.net.

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denotes the p-component of the group G. $Z(p^\infty)$ denotes the Prüfer group. It may be realised as the p-component of Q/Z. If one sets $a_n = 1/p^n + Z$ then the Prüfer group is defined by the relations $pa_1 = 0$, $pa_{n+1} = a_n$. The subgroup generated by a_n is isomorphic to $Z(p^n)$ and these are the only proper subgroups.

2. Survey

The conjecture of de Bruijn referred to above is that if G is a finite cyclic group and $G = A+B$ is a factorization in which one of the factors has prime order then either A or B is periodic. It follows from above that a periodic factor of prime order must be a subgroup. Several generalizations of this conjecture have been shown to hold. We shall list these results and then use the clever constructions of de Bruijn [4] to show that in a certain sense these results are best possible.

The first generalization was given in [6]. In Theorem 2 it is shown that the desired result holds if one of the two factors has prime power order.

The next generalization is given in [8]. It is shown in Theorem 2 that if $G = A_1 + \cdots + A_k + B$, where G is a finite cyclic group and each factor A_i has order equal to the power of a fixed prime p, then one of the factors is periodic.

In [9] it is shown that in Theorems 1 and 2 the above results still hold where it is no longer assumed that the group G is cyclic but only that the finite Abelian group G has cyclic p-component, where p is the prime referred to above.

We now outline the sense in which these results are best possible. We use Theorem 1 of de Bruijn [4] to construct an example. Let p, q be distinct primes and let G be a cyclic group of order p^3q^2. Then we choose H to be the subgroup of order p^2q^2 and then the subgroups H_1, H_2 are of orders p^2 and q^2 and their subgroups $H_1{}^*$ and $H_2{}^*$ have orders p and q respectively. Here we are using the notation of de Bruijn. His construction now gives subsets A and B which satisfy the factorization $G = A + B$ but which are not periodic. The subset A is the direct sum of two smaller non-periodic subsets D and E. Thus we have a bad factorization $G = B + D + E$ where the order $|B|$ of B is p^2q and $|D| = p$, $|E| = q$. So we see that the theorems will not allow two factors of distinct prime orders to exist if the desired result is to be obtained.

If we weaken the assumption in order to permit non-cyclic finite Abelian groups to be considered then again a construction of de Bruijn in Theorem 2 of [4] shows that the desired result fails to hold. For any prime $p > 3$ it is shown that if $G = Z(p) + Z(p^2)$ then there is a bad factorization $G = A + B$ where $|A| = p^2$ and $|B| = p$. In the case $p = 3$ we use Theorem 3 of de Bruijn [4]. This gives a bad factorization $G = A + B$ where $G = Z(9) + Z(9)$ and $|A| = |B| = 9$. In the case $p = 2$ Theorem 1 of [4] leads to a bad factorization $G = A + B + C$, where $G = Z(8) + Z(4)$ and $|A| = 8$, $|B| = |C| = 2$. Thus the results do not hold in general for non-cyclic finite Abelian groups.

There is one case where a family of finite non-cyclic groups satisfies the desired conclusion. This is the family of groups $Z(2^n) + Z(2)$. In the case of two factors this is shown in [7, Theorem 7] and in the cases of any number of factors this is shown in [1, Theorem 10]. We should point out however that unlike the cyclic case where $G = Z(p^n) + Z(q)$ has the desired property as all factors except one must be powers of p, the result does not extend to the non-cyclic groups $G = Z(2^n) + Z(2) + Z(q)$, where q is an odd prime and $n > 2$. This follows from Theorem 1 of de Bruijn [4]. As above we obtain a bad factorization $G = A + B + C$, where $|A| = 2^n$, $|B| = 2$, $|C| = q$. This follows by choosing $H = Z(2^{n-1}) + Z(2) + Z(q)$ and then $H_1 = Z(2^{n-1})$, $H_2 = Z(2) + Z(q)$.