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Words with a generalized restricted growth property

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Dedicated to the memory of N. G. de Bruijn

Abstract

Words where each new letter (natural number) can never be too large, compared to the ones that were seen already, are enumerated. The letters follow the geometric distribution. Also, the maximal letter in such words is studied. The asymptotic answers involve small periodic oscillations. The methods include a chain of techniques: exponential generating function, Poisson generating function, Mellin transform, depoissonization.

Keywords: Random words; Restricted growth property; Depoissonization; Mellin transform

1. Introduction

When Knuth started his fundamental series of books *The Art of Computer Programming* [8], de Bruijn was (one of) his asymptotic advisor(s). In particular, he suggested how to evaluate sums like

$$\sum_{k\geq 1}\left(1-e^{-n/2^k}\right) \quad \text{and} \quad \sum_{k\geq 1} d(k)e^{-k^2/n},$$

where $d(k)$ is the number of divisors of k. Although the word was not mentioned in the first editions, in essence it was the *Mellin transform* that found its way into [9]. Around the same time, the paper [2] appeared, which has 166 citations by google scholar.[1] This paper has a third

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coauthor, Rice, who also suggested asymptotic methods to Knuth; there is an innocent exercise in [9], which lead later to developments called *Rice's method*; see [4].

We briefly review the Mellin transform method in asymptotic enumeration, compare with [3,5].

$$\mathscr{M}[f(x);s] = f^*(s) = \int_0^\infty f(x)x^{s-1}dx.$$

There is the *harmonic sum property*

$$\mathscr{M}\left[\sum_{k\geq1} a_k f(b_k x); s\right] = \sum_{k\geq1} a_k b_k^{-s} \cdot \mathscr{M}[f(x);s]. \tag{1}$$

This is particularly useful if the series has a convergent closed form evaluation (often in terms of the zeta function etc.).

Typically the Mellin transform exists in a *vertical strip* of the complex plane. There is an *inversion formula*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds,$$

where c must be in the vertical strip. Shifting the line of integration to the left/right and collecting residues provides the asymptotic expansion. The choice of left/right depends on whether one needs the expansion for $x \to \infty$ or $x \to 0$; see the converse mapping theorem in [3] for a precise statement of this fact.

The most prominent example is $f(x) = e^{-x}$, so that $f^*(s) = \Gamma(s)$, whence the term *Gamma function method* was originally coined. During the last 40 years, de Bruijn's suggestion led to numerous further developments and applications.

In the technical part of this paper, we will indeed use the Mellin transform to deal with a combinatorial (discrete probability) problem. As often in combinatorics, the problem is not difficult to describe, although the solution requires some technical machinery. We consider words $w_1 w_2 \cdots w_n$ where the letters are positive integers, and integer k appears with (geometric) probability pq^{k-1}, and $p+q=1$. The letters are independent from each other. The *restricted growth property* is satisfied when

$$w_k \leq 1 + \max\{w_1, \ldots, w_{k-1}\} \quad \text{for all } k \text{ and } w_0 = 0.$$

The words that satisfy the restricted growth property are related to *set partitions* and *approximate counting* [10,12]. The asymptotic enumeration of restricted words of length n and the asymptotic study of $\max\{w_1, \ldots, w_n\}$ was done in [11,12] using the above mentioned Rice method.

Now it is a natural extension to introduce a parameter:

$$w_k \leq d + \max\{w_1, \ldots, w_{k-1}\} \quad \text{for all } k \text{ and } w_0 = 0. \tag{2}$$

For $d \geq 2$, the asymptotic problems are of a more delicate nature, and that is what we will do here. Rice's method is based on explicit enumerations represented as *alternating sums*. We use here a combination of techniques that is more flexible: poissonization/depoissonization and Mellin transform. Poissonization is the process of replacing the fixed n by a random variable which is Poisson distributed with parameter z; depoissonization is the reversed process that allows to go back from z to n. Typically, if $f(z)$ is an exponential generating function of a