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Isomorphism is equality

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Abstract

The setting of this work is dependent type theory extended with the univalence axiom. We prove that, for a large class of algebraic structures, isomorphic instances of a structure are equal—in fact, isomorphism is in bijective correspondence with equality. The class of structures includes monoids whose underlying types are "sets", and also posets where the underlying types are sets and the ordering relations are pointwise "propositional". For monoids on sets equality coincides with the usual notion of isomorphism from universal algebra, and for posets of the kind mentioned above equality coincides with order isomorphism.

Keywords: Dependent type theory; Proof assistants; Univalence

1. Introduction

De Bruijn argued that it is more natural for mathematicians to work with a typed language than with the untyped universe of set theory [6]. In this paper we explore a possible *mathematical* advantage of working in a type theory—inspired by the ones designed by de Bruijn and his coworkers[1] [7]—over working in set theory.

Consider the following two monoids:

$$(\mathbb{N}, \lambda mn.\, m + n, 0)$$

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[1] The AUTOMATH project team included van Benthem Jutting, van Daalen, Kornaat, Nederpelt, de Vrijer, Zandleven, Zucker, and others [13].

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and

$$(\mathbb{N} \setminus \{ 0 \}, \lambda mn.\, m + n - 1, 1).$$

These monoids are *isomorphic*, as witnessed by the isomorphism $\lambda n.\ n + 1$. However, in set theory they are not *equal*: there are properties that are satisfied by only one of them. For instance, only the first one satisfies the property that the carrier set contains the element 0.

In (a certain) type theory extended with the univalence axiom (see Section 2) the situation is different. This is the focus of the present paper:

- We prove that monoids M_1 and M_2 that are isomorphic, i.e. for which there is a homomorphic bijection $f : M_1 \to M_2$, are equal (see Section 3.5). In fact, we show that isomorphism is in bijective correspondence with equality.

  Note that the equality that we use is substitutive. This means that, unlike in set theory, any property that holds for the first monoid above also holds for the second one.

  (The result is restricted to monoids whose carrier types are "sets". This term is defined in Section 2.5. Many types, including the natural numbers, are sets.)
- The result about monoids follows directly from a more general theorem (see Section 3.3), which applies to a large class of algebraic structures, including posets and discrete fields (defined as in Section 3.5).

All the main results in the paper have been formalised using the proof assistant Agda[2] [15,17], which is based on Martin-Löf type theory [12,14]. Unlike in regular Martin-Löf type theory we use a "non-computing" J rule (i.e. the computation rule for J only holds propositionally, not definitionally); this choice, which makes the result more generally applicable, is motivated in Section 2.3. We believe that our arguments carry over to other variants of type theory, but do not make any formal claims in this direction.

Note that our theorem is proved *inside* the type theory, using the univalence axiom. In the absence of this axiom we can still observe, *meta-theoretically*, that we cannot prove any statement that distinguishes the two monoids above (given the consistency of the axiom). A related observation was made already in the 1930s by Lindenbaum and Tarski [10] (see also [16]): in a certain variant of type theory every sentential function is invariant under bijections.

The formulation of "isomorphism is equality" that is used in this paper is not intended to be as general as possible; we try to strike a good balance between generality and ease of understanding. Other variations of this result have been developed concurrently by Aczel [18] and Ahrens et al. [1]. See Section 4 for further discussion of related work.

2. Preliminaries

This section introduces some concepts, terminology and results used below. We assume some familiarity with type theory.

The presentation in this and subsequent sections is close to the Agda formalisation, but differs in minor details. In particular, we do not always use proper Agda syntax.

[2] Using the `--without-K` flag; the code has been made available to download.