Available online at www.sciencedirect.com

SciVerse ScienceDirect

Indagationes Mathematicae 22 (2011) 165–206

indagationes mathematicae

www.elsevier.com/locate/indag

Cyclicity of common slow–fast cycles

P. De Maesschalck[a], F. Dumortier[a,*], R. Roussarie[b]

[a] *Hasselt University, Campus Diepenbeek, Agoralaan gebouw D, B-3590 Diepenbeek, Belgium*
[b] *Institut de Mathématique de Bourgogne, U.M.R. 5584 du C.N.R.S., Université de Bourgogne, B.P. 47 870, 21078 Dijon Cedex, France*

Abstract

We study the limit cycles of planar slow–fast vector fields, appearing near a given slow–fast cycle, formed by an arbitrary sequence of slow parts and fast parts, and where the slow parts can meet the fast parts in a nilpotent contact point of arbitrary order. Using the notion slow divergence integral, we delimit a large subclass of these slow–fast cycles out of which at most one limit cycle can perturb, and a smaller subclass out of which exactly one limit cycle will perturb. Though the focus lies on common slow–fast cycles, i.e. cycles with only attracting or only repelling slow parts, we present results that are valid for more general slow–fast cycles. We also provide examples of attracting common slow–fast cycles out of which more than one limit cycle can perturb, one of which is repelling.

Keywords: Slow–fast cycle; Cyclicity; Contact point; Singular perturbations; Canard; Blow-up; Relaxation oscillation

1. Introduction

This paper is devoted to the study of limit cycles appearing in singularly perturbed families of planar vector fields (or more generally vector fields X_ϵ on an orientable 2-manifold). We consider an ϵ-family of vector fields on a 2-manifold that, for $\epsilon = 0$, has a curve of singular points. Such a curve will be called a slow curve and will be denoted S. In general, S consists of hyperbolically attracting points, hyperbolically repelling points and contact points, depending on whether the linear part of the vector field at that point of S has a negative nonzero eigenvalue, a

* Corresponding author. Tel.: +32 11 26 82 39; fax: +32 11 26 82 99.
E-mail addresses: peter.demaesschalck@uhasselt.be (P. De Maesschalck), freddy.dumortier@uhasselt.be (F. Dumortier), roussari@u-bourgogne.fr (R. Roussarie).

doi:10.1016/j.indag.2011.09.008

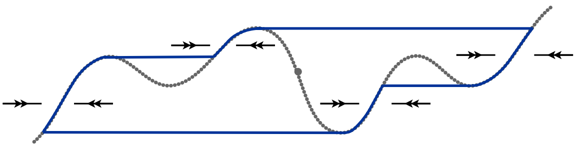

Fig. 1. An example of a common slow–fast cycle.

positive nonzero eigenvalue, or 2 zero eigenvalues. In this paper, we will only consider contact points of nilpotent type, subject to some mild extra conditions (see later). The main question that we deal with is to describe the dynamics near a so-called *slow–fast cycle*, focusing first on *common slow–fast cycles*. In particular we study the limit cycles that can perturb from such a slow–fast cycle. Let us explain the notion "slow–fast cycle" and "common slow–fast cycle". It is well-known that orbits of X_ϵ, for $\epsilon > 0$ small, are Hausdorff close to a succession of fast orbits and parts of the slow curve. Fast orbits are regular orbits of X_0. The presence of a curve of singular points of X_0 typically reduces the complexity of the phase portrait: points near the curve are either attracted or repelled away from the slow curve. In the neighborhood of the slow curve, the behavior of X_ϵ, for $\epsilon > 0$ small, is determined by a slow drift along the slow curve. This is called the *slow dynamics*.

A *slow–fast cycle* is a succession of slow parts and fast orbits, such that the end points coincide. For $\epsilon = 0$ we of course have no limit cycles; limit cycles may exist for $\epsilon > 0$ small, bifurcating from a slow–fast cycle. A *common slow–fast cycle* is a slow–fast cycle where the slow parts are either all attracting or all repelling, see for example in Fig. 1. This notion is seen in contrast with a canard cycle, which is a slow–fast cycle where both attracting and repelling slow parts are present. In this paper, we obtain results on both common and canard-type slow–fast cycles in a quite general setting, but to present the ideas, *we will focus in the introduction on attracting common slow–fast cycles* (the case of repelling common cycles is completely similar by inversion of time). Furthermore, we will first suppose that the slow dynamics is regular as well outside as at the contact points. If this is the case, we speak of a regular common slow–fast cycle. Later in the paper we will obtain a result for a more general class of slow–fast cycles.

There are two main results in this paper. On one hand we prove that one can have at most one limit cycle perturbing from an attracting common slow–fast cycle Γ; on the other hand we show the occurrence of a (unique) limit cycle of X_ϵ when perturbing from a *strongly* common slow–fast cycle. For the first result (that in fact will be proven to hold for a slightly more general type of slow–fast cycles than the attracting common one), we essentially study the first return map along Γ, and prove that it has a derivative strictly between 0 by 1 (for $\epsilon > 0$). To that end, we relate this derivative of the first return map to the exponential of the integral of the divergence of the vector field. We prove that this divergence integral is dominated by the contributions it has from the slow parts (i.e. the fast parts are negligible, and so are the parts in the vicinity of contact points). Since the slow parts of a common cycle are all attracting or all repelling, the divergence integral surely has a fixed sign, from which the required results on the derivative of the first return map are derived. Such a bound on the divergence integral will be obtained for arbitrary regular common slow–fast cycles, showing that at most one limit cycle may perturb from it. We then generalize the result to a slightly larger class of slow–fast cycles.

On the other hand, we will introduce the notion "strongly common slow–fast cycle", and for this subset of common cycles, we will show that necessarily a (unique) limit cycle perturbs from it. The existence part in this statement will be done by covering the slow–fast cycle with a chain of limits of flow box neighborhoods. More details will be given in Sections 5 and 6.