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Indagationes Mathematicae 24 (2013) 373–381

indagationes mathematicae

www.elsevier.com/locate/indag

One-sided power sum and cosine inequalities

Frits Beukers[a], Rob Tijdeman[b,*]

[a] *Mathematical Institute, University Utrecht, P.O.Box 80010, 3508 TA Utrecht, The Netherlands*
[b] *Mathematical Institute, Leiden University, P.O.Box 9512, 2300 RA Leiden, The Netherlands*

Received 13 January 2012; received in revised form 6 November 2012; accepted 29 November 2012

Communicated by J. Top

Abstract

In this note we prove results of the following types. Let be given distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and for every z_j there exists an i such that $z_i = \overline{z_j}$. Then

$$\inf_k \sum_{j=1}^{n} z_j^k \leq -1.$$

If, moreover, none of the ratios z_i/z_j with $i \neq j$ is a root of unity, then

$$\inf_k \sum_{j=1}^{n} z_j^k \leq -\frac{1}{\pi^4} \log n.$$

The constant -1 in the former result is the best possible. The above results are special cases of upper bounds for $\inf_k \sum_{j=1}^{n} b_j z_j^k$ obtained in this paper.

Keywords: Power sums; One-sided inequality; Sums of cosines; Littlewood conjecture; Multistep methods

1. Introduction

Our colleague Marc N. Spijker asked the following question in view of an application in numerical analysis [6]:

* Corresponding author. Tel.: +31 715893046; fax: +31 715277101.
E-mail addresses: f.beukers@uu.nl (F. Beukers), tijdeman@math.leidenuniv.nl (R. Tijdeman).

doi:10.1016/j.indag.2012.11.009

Problem 1. Is it true that for given real numbers $b_j \geq 1$ and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and

for every z_j there exists an i such that $z_i = \overline{z_j}, b_i = b_j$

we have $\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k \leq -1$?

Note that by the conjugacy conditions on b_i, z_i the sum $\sum_{j=1}^n b_j z_j^k$ is real for all k.

In Section 2 we answer Spijker's question in a slightly generalized and sharpened form (see Theorem 1 and Corollary 1). The solution of Problem 1 has an application to numerical analysis, more particularly Linear multistep methods (LMMs). They form a well-known class of numerical step-by-step methods for solving initial-value problems for certain systems of ordinary differential equations. In many applications of such methods it is essential that the LMM has specific stability properties. An important property of this kind is named *boundedness* and has recently been studied by Hundsdorfer, Mozartova and Spijker [3]. In that paper the *stepsize-coefficient* γ is a crucial parameter in the study of boundedness. In [6] Spijker attempts to single out all LMMs with a positive stepsize-coefficient γ for boundedness. By using Corollary 1 below he is able to nicely narrow the class of such LMMs.

As a fine point we can remark that the bound -1 in Spijker's problem is the optimal one. Namely, take $z_j = \zeta^j$ where $\zeta = e^{2\pi i/(n+1)}$ and $b_j = 1$ for all j. Then the exponential sum equals n if k is divisible by $n+1$ and -1 if not.

If, moreover, none of z_j/z_i with $i \neq j$ is a root of unity, then the upper bound in Problem 1 can be improved to $-\log n/\pi^4$. We deal with this question in Theorem 3 and more particularly Corollary 2. The obtained results can easily be transformed into estimates for $\inf_{k\in\mathbb{Z}} \sum_{j=1}^m b_j \cos(2\pi\alpha_j k)$ where b_j, α_j are real numbers and $\alpha_1, \ldots, \alpha_n$ are strictly between 0 and 1/2. Theorem 4 states that this infimum is equal to $\inf_{t\in\mathbb{R}} \sum_{j=1}^m b_j \cos(2\pi\alpha_j t)$, provided that the $\mathbb{Q}$-span of $\alpha_1, \ldots, \alpha_n$ does not contain 1.

2. The general case

We provide an answer to Problem 1.

Theorem 1. *Let n be a positive integer. Let $b_1, \ldots, b_n$ be nonzero complex numbers such that $b_{n+1-i} = \overline{b_i}$ for all $i = 1, 2, \ldots, n$. Let $z_1, \ldots, z_n$ be distinct complex numbers with absolute value 1, not equal to 1, such that $z_{n+1-i} = \overline{z_i}$ for all $i = 1, 2, \ldots, n$. Then*

$$\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k \leq -\frac{\sum_{j=1}^n |b_j|^2}{\sum_{j=1}^n |b_j|}.$$

Note that $\sum_{j=1}^n b_j z_j^k$ is real because of the conjugacy conditions.

By applying the Cauchy–Schwarz inequality we immediately obtain the following consequence.

Corollary 1. *Let n, b_j, z_j be as in Theorem 1. Then*

$$\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k \leq -\frac{1}{n} \sum_{j=1}^n |b_j|.$$