Available online at www.sciencedirect.com

SciVerse ScienceDirect

Indagationes Mathematicae 24 (2013) 382–390

indagationes mathematicae

www.elsevier.com/locate/indag

On Gauss problem for the Lüroth expansion

Marius Iosifescu[a], Gabriela Ileana Sebe[b,*]

[a] *Romanian Academy, "Gheorghe Mihoc-Caius Iacob", Institute for Mathematical Statistics and Applied Mathematics, Calea 13 Septembrie 13, 050711 Bucharest, Romania*

[b] *"Politehnica" University of Bucharest, Faculty of Applied Sciences, Splaiul Independenţei 313, 060042, Bucharest, Romania*

Received 25 August 2012; received in revised form 29 November 2012; accepted 10 December 2012

Communicated by R. Tijdeman

Abstract

Consider the transformation $\tau(x) = \left[\frac{1}{x}\right]\left(\left(\left[\frac{1}{x}\right]+1\right)x-1\right)$, $x \neq 0$, $\tau(0) = 0$, of $I = [0,1]$ which underlines the Lüroth expansion. Let $\mu \ll \lambda$ (Lebesgue measure on $[0,1]$). We show that $\mu\tau^{-n}(A)$ approaches $\lambda(A)$ uniformly in $A \in \mathcal{B}_I$ with reminder $O\left(q^n\right)$, $0 < q < 1$, as $n \to \infty$, for certain densities $\mathrm{d}\mu/\mathrm{d}\lambda$.

© 2012 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.

Keywords: Lüroth series; Random system with complete connections; Perron–Frobenius operator; Piecewise monotonic transformation

1. Introduction

Lüroth [7] introduced and studied the following series expansion. Let $x \in I$. Then

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)} + \cdots + \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)} + \cdots, \tag{1}$$

* Corresponding author. Tel.: +40 21 4029489; fax: +40 21 3181011.
E-mail addresses: miosifes@acad.ro (M. Iosifescu), igsebe@yahoo.com (G.I. Sebe).

0019-3577/$ - see front matter © 2012 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.indag.2012.12.003

where the *digits* (*incomplete quotients*) $d_n(x), n \in \mathbb{N}_+ = \{1, 2, \ldots\}$, are natural integers ≥ 2. (We will suppress x in the notation when confusion is not possible.) Lüroth showed that every irrational number has a unique infinite expansion (1) and that each rational either has a finite or an infinite periodic expansion. The series expansion (1) is called the *Lüroth series* of x.

We note that the Lüroth expansion is generated by the transformation $\tau : I \to I$ defined as

$$\tau(x) = \left[\frac{1}{x}\right]\left(\left(\left[\frac{1}{x}\right] + 1\right)x - 1\right), \quad x \neq 0, \qquad \tau(0) = 0. \tag{2}$$

Here $[\cdot]$ stands for the integer part. For $x \in I$ we define $d_1(x) = \left[\frac{1}{x}\right] + 1, x \neq 0; d_1(0) = \infty$ and $d_{n+1}(x) = d_1(\tau^n(x)), n \in \mathbb{N}_+$. It follows from (2) that $\tau(x) = d_1(x)(d_1(x) - 1)x - (d_1(x) - 1)$ and, therefore,

$$\begin{aligned} x &= \frac{1}{d_1(x)} + \frac{\tau(x)}{d_1(x)(d_1(x)-1)} \\ &= \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)}\left(\frac{1}{d_1(\tau(x))} + \frac{\tau^2(x)}{d_1(\tau(x))(d_1(\tau(x)-1))}\right) \\ &= \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)} \\ &\quad + \frac{\tau^2(x)}{d_1(x)(d_1(x)-1)d_2(x)(d_2(x)-1)} = \cdots \\ &= \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)} \\ &\quad + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)(d_2(x)-1)d_3(x)} + \cdots \\ &\quad + \cdots + \frac{\tau^n(x)}{d_1(x)(d_1(x)-1)\cdots d_n(x)(d_n(x)-1)}. \end{aligned}$$

Putting

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{d_1(x)} + \sum_{k=1}^{n-1} \frac{1}{d_1(x)(d_1(x)-1)\cdots d_k(x)(d_k(x)-1)d_{k+1}(x)}, \quad n \geq 1, \tag{3}$$

where $q_1(x) = d_1(x), q_n(x) = d_1(x)(d_1(x)-1)\cdots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x), n \geq 2$, *the* n-th *convergent of* x, it follows from (3) that

$$x - \frac{p_n(x)}{q_n(x)} = \frac{\tau^n(x)}{q_n(x)(d_n(x)-1)}, \quad n \geq 1.$$

It follows from $d_n \geq 2$ and $0 \leq \tau^n \leq 1$ that the sum (1) converges to x. We will write $x = (d_1(x), d_2(x), \ldots)$ and $\frac{p_n}{q_n} = (d_1, d_2 \ldots, d_n)$.

In addition we should mention that the digits $d_n, n \in \mathbb{N}_+$, may be considered as random variables on I equipped with the σ-algebra $\mathcal{B}_I$ of all Borel subsets of I. These are almost

Download English Version:

https://daneshyari.com/en/article/4673097

Download Persian Version:

https://daneshyari.com/article/4673097

Daneshyari.com