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Bounds for discrete tomography solutions

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Abstract

We consider the reconstruction of a function on a finite subset of $\mathbb{Z}^2$ where the line sums in certain directions are prescribed. Its real solutions form a linear manifold, its integer solutions a grid. First we provide an explicit expression for the projection vector from the origin onto the linear solution manifold in the case of only row and column sums of a finite subset of $\mathbb{Z}^2$. Next we present a method for estimating the maximal distance between two binary solutions. Subsequently we deduce an upper bound for the distance from any given real solution to the nearest integer solution. This enables us to estimate the stability of solutions. Finally we generalize the first result mentioned above to the continuous case.

Keywords: Discrete tomography; Upper bounds; Approximate solutions; Stability; Projection vector

1. Introduction

The basic problem of discrete tomography is that of reconstructing a function $f : A \to B$ where A is a finite subset of $\mathbb{Z}^l$ and B a finite subset of $\mathbb{R}$ for the case where the sums of the function values along the lines in a finite number of directions are given. In this paper we consider $l = 2$. There is a vast literature on the special case $A = \{(i, j) \in \mathbb{Z}^2 | 0 \leq i < m, 0 \leq j < n\}, B = \{0, 1\}$, where the problem is that of finding the function values

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from the given row and column sums. In 1957 Ryser [23] and Gale [17] independently derived necessary and sufficient conditions for the existence of a solution in this special case. Ryser also provided a polynomial time algorithm for finding such a solution. However, the problem is usually highly underdetermined and a large number of solutions may exist [25]. Therefore the quest is often to find a solution of a certain type. For some classes of highly structured images, such as hv-convex polyominoes (the 1's in each row and column are contiguous), polynomial time reconstruction algorithms have been developed (see e.g. [5,11,12]). Woeginger [26] presented an overview of classes of polyominoes for which it is proved that either they can be reconstructed in polynomial time or reconstruction is NP-hard. Batenburg [6] developed an evolutionary algorithm for finding the reconstruction which maximizes an evolution function and showed that the algorithm can be successfully applied to a wide range of evolution functions.

We consider solutions as vectors with the values of f as entries. Hajdu and Tijdeman [19] observed that the set of binary solutions is precisely the set of shortest vector solutions in the set of functions $f : A \to \mathbb{Z}$ with the given line sums, provided that such solutions exist. They also showed that the solutions $f : A \to \mathbb{Z}$ with the given line sums form a multidimensional grid on the linear manifold which consists of all the solutions $f : A \to \mathbb{R}$ with the given line sums. Moreover, they determined the dimension of this manifold and indicated how to find a set of generators of the grid. Later they used their analysis to develop an algorithm for actually constructing solutions $f : A \to \{0, 1\}$ in [20], whereafter Batenburg [7,8] constructed much faster algorithms.

For functions $f : \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\} \to \mathbb{R}$ and given row and column sums we deduce a new and explicit expression for the projection vector $\vec{f_0}$ from the origin onto the real linear solution manifold in Section 3. We do not know of a similar expression for other sets of line sums, but show that the result can be extended to the continuous case ($A = [0, m] \times [0, n] \subset \mathbb{R}^2$) in Section 6. In this section we answer a question posed by Joost Batenburg.

In many applications it suffices to find a solution or almost-solution which is guaranteed to be similar to the original. If all the solutions are similar, then they will also be similar to the original and we say that the solution set is stable. Alpers, Gritzmann and Thorens [4] showed that just a small change in the data can lead to a dramatic change in the image. Research on uniqueness and stability of solutions was carried out by Alpers et al. [1–3] and van Dalen [13–16] for cases where only row sums and column sums are given. Their estimates depend on only a few parameters. More general situations were studied by Brunetti and Daurat [10] and by Gritzmann, Langfeld and Wiegelmann [18]. In Section 4 we use the distance estimates for the solutions to derive new stability results. They involve more parameters, but, at least in the given example, yield better results than the estimates obtained by van Dalen.

In Stolk [24] a system of line sums is called compatible if and only if a real solution exists. (Then the projection vector $\vec{f_0}$ is an example of such a real solution.) Stolk showed that if the line sums are integers and they are compatible, then there exists an integer solution. In Section 5 we derive an upper bound for the Euclidean distance from $\vec{f_0}$ to the nearest integer solution.

The interest in discrete tomography arose from the study of atom positions in a crystal, but the theory developed also has applications in medical imaging and in nuclear science; see [21,22]. The results in the present paper are merely of theoretical interest, but applications of the method can be found in [9] and they may help with estimating how many directions are needed to be sure that the solution is unique and hence that one is certain of having found the original configuration.