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Closure properties associated to natural equivalences

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Abstract

Given a pair of adjoint functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$, we study some closure properties of some full subcategories $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ such that the restrictions $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ induce an equivalence.

Keywords: Adjoint functors; Static object; Natural equivalence; Grothendieck category

1. Introduction

The study of equivalences induced by pairs of adjoint covariant functors is an important topic. Using Hom and $\otimes$ functors, this topic developed important concepts in the Module Theory. These concepts, as tilting and star module, are used in the study of representable equivalences (see [12,26] for complete surveys on the subjects). Moreover, this kind of study is also useful in a more general setting, in order to apply the results to other kind of categories. For instance, Castaño-Iglesias, Gómez-Torrecillas and Wisbauer applied the study of adjoint pairs of functors between Grothendieck categories to special categories of graded modules or comodules [11]. Marcus and Modoi, in [20], also used other kind of equivalences in order to study categories of graded modules. Colpi [13], Gregorio [18] and Rump [24] constructed a general theory of tilting objects in various kind of categories. Recently, Bazzoni [5] considers some particular categories of fractions (which, in general, have no infinite direct sums) in order to describe the

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classes involved in a tilting theorem [5, Theorem 4.5], while Breaz [8,6] studied functors and equivalences between similar categories of fractions in order to apply these results to the category of abelian groups and quasi-homomorphisms (see [1] for such an application).

In the study of equivalences induced by a pair of adjoint functors, some closure properties of the classes involved in these equivalences are very important. For instance, some closure properties with respect to submodules or to quotients are used to characterize the representability of these functors [15,21] and the pairs of functors which generalize tilting modules and star modules can be characterized by some closure properties [14,24]. Furthermore, closure properties of the classes of static modules, respectively adstatic modules (i.e. the maximal classes such that the maps of adjunction are isomorphisms), with respect to submodules or to submodules of finite index are very useful in the study of abelian groups which have some flatness properties as modules over their endomorphism ring [2,7] or of the important class of S-groups [4] and their generalizations [3].

In this paper we study some closure properties which are inspired from the work of Mantese and Tonolo (see [19]) and Fuller (see [16]) on dualities. We start with a preliminary section where we present some basic notions and results. Then, in Section 3, which is the main section of the paper, we consider a pair of adjoint additive covariant functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ between two abelian categories. We are interested about the closure properties of some full subcategories $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ such that the restrictions $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ are equivalences. More precisely, we characterize situations when $\overline{\mathcal{B}}$ is closed with respect to faithful factors by a closure property of $\overline{\mathcal{A}}$ and an exactness property of F in Theorem 3.3 and Proposition 3.7. Then, in Proposition 3.8, we identify when the converse of the exactness property of F is valid. These results are applied in Section 4 for closure properties of some classes constructed starting with the class $\mathrm{add}(V)$, where $V = \mathrm{F}(U)$ and U is a static object. We recall that, for an object X, $\mathrm{add}(X)$ denotes the class of all direct summands of finite direct sums of copies of X.

If R is an unital associative ring, we denote by Mod-R (respectively, by R-Mod) the category of all right (respectively, left) R-modules. As in the papers [10,11,23], the results obtained here can be applied to the pairs of adjoint additive and covariant functors presented in Section 2.

2. Preliminaries

Throughout this paper, we consider a pair of additive and covariant functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ between abelian categories such that G is a left adjoint to F, i.e. there are natural isomorphisms

$$\varphi_{X,M} : \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}(X), M) \to \mathrm{Hom}_{\mathcal{B}}(X, \mathrm{F}(M)),$$

for all $M \in \mathcal{A}$ and for all $X \in \mathcal{B}$. Then, they induce two natural transformations

$$\phi : \mathrm{GF} \to 1_{\mathcal{A}}, \quad \text{defined by } \phi_M = \varphi^{-1}_{\mathrm{F}(M),M}(1_{\mathrm{F}(M)})$$

and

$$\theta : 1_{\mathcal{B}} \to \mathrm{FG}, \quad \text{defined by } \theta_X = \varphi_{X,\mathrm{G}(X)}(1_{\mathrm{G}(X)}).$$

We note that F is left exact and G is right exact. Moreover, they satisfy the identities $\mathrm{F}(\phi_M) \circ \theta_{\mathrm{F}(M)} = 1_{\mathrm{F}(M)}$ and $\phi_{\mathrm{G}(X)} \circ \mathrm{G}(\theta_X) = 1_{\mathrm{G}(X)}$, for all $M \in \mathcal{A}$ and for all $X \in \mathcal{B}$. We also assume that all considered subcategories are isomorphically closed.

The classical example of such a pair of functors is the following (see [25], Chapter 9):