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The Gauss map of a harmonic surface

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Abstract

We prove that the distortion function of the Gauss map of a surface parametrized by harmonic coordinates coincides with the distortion function of the parametrization. Consequently, the Gauss map of a harmonic surface is $\mathcal{K}$ quasiconformal if and only if its harmonic parametrization is $\mathcal{K}$ quasiconformal, provided that the Gauss map is regular or what is shown to be the same, provided that the surface is non-planar. This generalizes the classical result that the Gauss map of a minimal surface is a conformal mapping.

Keywords: Gauss map; Quasiconformal harmonic surfaces; Minimal surfaces; Regular surfaces

1. Introduction and statement of the main result

Let (N, h) be a Riemannian manifold and (M, g) be a compact Riemannian manifold. A smooth map $X : M \to N$ between M and N is said to be *harmonic* if it is a critical point of the Dirichlet functional $E[X] := \int_M \|dX\|^2 dV_g$. This definition is extended to the case where M is not compact by requiring the restriction of X to every compact domain to be harmonic, or, more typically, requiring that X be a critical point of the energy functional in the Sobolev space $H_2^1(M, N)$.

Equivalently, the map X is harmonic if it satisfies the Euler–Lagrange equations associated to the functional E. Harmonic maps model the extrema of the energy functionals associated to some physical phenomena in viscosity, dynamic of fluids, electromagnetism, cosmology.... When

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$N = \mathbf{R}^n$, the Euler–Lagrange equation is indeed the Dirichlet equation $\Delta X = 0$, where Δ is the Laplace operator on M. See, for instance, the surveys [6] and the references therein for a good setting.

If M has dimension two, the Dirichlet functional actually depends only on the conformal structure of the surface. The interplay between conformality, minimality and harmonicity is a core topic in geometrical analysis, which connects the complex analysis and differential geometry. It is well known that a conformal immersion of a Riemann surface in the Euclidean three-space is minimal if and only if it is harmonic, and in this case its Gauss map is conformal (meromorphic). This paper is motivated by the recent paper of Alarcon and Lopez [2]. In that paper the authors considered parametric harmonic quasiconformal immersions of surfaces in the sense of Definition 1.1 into the Euclidean space $\mathbf{R}^3$ and prove some interesting facts. A typical result from [2] is the following: A complete harmonic immersion X of an open Riemann surface M into $\mathbf{R}^3$ has finite total curvature if and only if it is algebraic and quasiconformal. In this paper we will consider the class of parametric harmonic quasiconformal surfaces in the sense of the standard definition of quasiconformality. Our approach is completely different from and complementary to the approach of Alarcon and Lopez. Among others, we will show that the definition of quasiconformality of harmonic parametrization of Alarcon and Lopez (Definition 1.1), coincides with the standard definition of quasiconformality of parametrization provided that the surface is not planar. On the other hand this result can be considered as an extension of a classical result for minimal surfaces: if the minimal surface is given by Enneper–Weierstrass parametrization, then its complex Gauss map is a meromorphic function.

1.1. Parametric and harmonic surfaces

We define an oriented parametric surface $\mathcal{M}$ in $\mathbf{R}^3$ to be an equivalence class of mappings $X = (a, b, c) : \Omega \to \mathbf{R}^3$ of some domain $\Omega \subset \mathbf{C}$ into $\mathbf{R}^3$, where the coordinate functions a, b, c are of class at least $C^1(\Omega)$. Two such mappings $X : \Omega \to \mathbf{R}^3$ and $\tilde{X} : \tilde{\Omega} \to \mathbf{R}^3$, referred to as parametrizations of the surface, are said to be equivalent if there is a C^1-diffeomorphism $\phi : \tilde{\Omega} \xrightarrow{\text{onto}} \Omega$ of positive Jacobian determinant such that $\tilde{X} = X \circ \phi$. Let us call such a diffeomorphism ϕ a *change of variables, or reparametrization* of the surface. Furthermore, we assume that the branch (critical) points of $\mathcal{M}$ are isolated. These are the points $(x, y) \in \Omega$ at which the tangent vectors $X_x = \frac{\partial Y}{\partial x}$, $X_y = \frac{\partial Y}{\partial y}$ are linearly dependent or equivalently $X_x \times X_y = 0$ where by $\times$ is denoted the standard vectorial product in the space $\mathbb{R}^3$. Equivalently, at the critical points the *Jacobian matrix* $\nabla X(z)$ has rank at most 1. It has full rank 2 at the *regular points*. A surface with no critical points is called an *immersion* or a *regular surface*. If a surface $\mathcal{M}$ allows a parametrization $X = (x_1, x_2, x_3) : \Omega \to \mathcal{M}$ satisfying the Laplace equation $\Delta X = (0, 0, 0)$, then the surface is called *harmonic surface*. We will simultaneously use the terminology harmonic surface for the surface $\mathcal{M}$ and for its harmonic parametrization X. The same surface $\mathcal{M}$ could admit several harmonic parametrizations. For example if Ω is the unit disk $\mathbf{U}$ and if $X : \mathbf{U} \to \mathcal{M}$ is a parametrization and φ is a Möbius transformation of the unit disk, then $f \circ \varphi$ is a harmonic parametrization. A *harmonic immersion* is a regular harmonic parametrization

1.2. Standard definition of quasiconformality

An orientation preserving smooth mapping $\varphi : \Omega \to \Omega'$, between two open domains $\Omega, \Omega' \subset \mathbf{C}$ is called $\mathcal{K}$ ($\mathcal{K} \geq 1$) quasiconformal (QC for short) if the dilatation d_f of the