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Brauer algebras of type F_4

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Abstract

We present an algebra related to the Coxeter group of type F_4 which can be viewed as the Brauer algebra of type F_4 and is obtained as a subalgebra of the Brauer algebra of type E_6. We also describe some properties of this algebra.

Keywords: Coxeter group; Brauer algebra; Root system

1. Introduction

When studying tensor decompositions for orthogonal groups, Brauer [2] introduced algebras which we now call Brauer algebras of type A. Cohen, Frenk and Wales [4] extended the definition to simply laced types, including type E_6. Tits [18] described how to obtain the Coxeter group of type F_4 as the fixed subgroup of the Coxeter group of type E_6 under a diagram automorphism (also seen in [3]). For this, Mühlherr gave a more general way by admissible partitions to obtain Coxeter groups as subgroups in Coxeter groups in [15]. Here we will apply a similar method to the Brauer algebra $\mathrm{Br}(E_6)$. This is a part of a project to define Brauer algebras of spherical types [7,6]. It turns out that the presentation by generators and relations obtainable from the Dynkin diagram of type F_4 is in the same way as was done for types B_n [6] and C_n [7].

First we give the definition of $\mathrm{Br}(F_4)$ using a presentation. Let δ be the generator of the infinite cyclic group.

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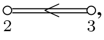
Definition 1.1. The Brauer algebra of type F_4, denoted by $\mathrm{Br}(\mathrm{F}_4)$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$-algebra generated by $\{r_i, e_i\}_{i=1}^4$, subject to the following relations.

$$r_i^2 = 1 \quad \text{for any } i \tag{1.1}$$
$$r_i e_i = e_i r_i = e_i \quad \text{for any } i \tag{1.2}$$
$$e_i^2 = \delta e_i \quad \text{for } i > 2 \tag{1.3}$$
$$e_i^2 = \delta^2 e_i \quad \text{for } i < 3 \tag{1.4}$$
$$r_i r_j = r_j r_i, \quad \text{for } i \nsim j \tag{1.5}$$
$$e_i r_j = r_j e_i, \quad \text{for } i \nsim j \tag{1.6}$$
$$e_i e_j = e_j e_i, \quad \text{for } i \nsim j \tag{1.7}$$
$$r_i r_j r_i = r_j r_i r_j, \quad \text{for } i \sim j \tag{1.8}$$
$$r_j r_i e_j = e_i e_j, \quad \text{for } i \sim j \tag{1.9}$$
$$r_i e_j r_i = r_j e_i r_j, \quad \text{for } i \sim j \tag{1.10}$$

and for 
,

$$r_2 r_3 r_2 r_3 = r_3 r_2 r_3 r_2 \tag{1.11}$$
$$r_2 r_3 e_2 = r_3 e_2 \tag{1.12}$$
$$r_2 e_3 r_2 e_3 = e_3 e_2 e_3 \tag{1.13}$$
$$(r_2 r_3 r_2) e_3 = e_3 (r_2 r_3 r_2) \tag{1.14}$$
$$e_2 r_3 e_2 = \delta e_2 \tag{1.15}$$
$$e_2 e_3 e_2 = \delta e_2 \tag{1.16}$$
$$e_2 r_3 r_2 = e_2 r_3 \tag{1.17}$$
$$e_2 e_3 r_2 = e_2 e_3. \tag{1.18}$$

Here $i \sim j$ means that i and j are connected by a simple bond and $i \nsim j$ means that there is no bond (simple or multiple) between i and j in the Dynkin Diagram of type F_4 depicted in the Fig. 1. The submonoid of the multiplicative monoid of $\mathrm{Br}(\mathrm{F}_4)$ generated by δ, $\{r_i, e_i\}_{i=1}^4$ is denoted by $\mathrm{BrM}(\mathrm{F}_4)$. This is the monoid of monomials in $\mathrm{Br}(\mathrm{F}_4)$.

Later, Lemma 2.1 implies that more relation can be obtained from the above defining relations.

The defining relations (1.11)–(1.18) can be found in $\mathrm{Br}(\mathrm{C}_2)$ in [7] and $\mathrm{Br}(\mathrm{B}_2)$ in [6] by renumbering indices. Note that these relations are not symmetric for 2 and 3. Their relations are fully determined by the Dynkin diagram in the sense that all relations depend only on the vertices and bonds of the Dynkin diagram and the lengths of their roots.

It is well known that the Coxeter group $W(\mathrm{F}_4)$ of type F_4, can be obtained as the subgroup from the Coxeter group $W(\mathrm{E}_6)$ of type E_6 ([18] or [3]), of elements invariant under the automorphism of $W(\mathrm{E}_6)$ determined by the diagram automorphism $\sigma = (1, 6)(3, 5)$ indicated as a permutation on the generators of $W(\mathrm{E}_6)$ whose Dynkin diagram are labeled and presented in Fig. 1.

The action σ can be extended to an automorphism of the Brauer algebra of type E_6 by acting on the Temperley–Lieb generators E_i [16] by the same permutation as for Weyl group generators. We denote by $\mathrm{SBr}(\mathrm{E}_6)$ the subalgebra generated by σ-invariant elements in $\mathrm{BrM}(\mathrm{E}_6)$. The main