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Deformations of Lie brackets and representations up to homotopy

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Abstract

We show that representations up to homotopy can be differentiated in a functorial way. A van Est type isomorphism theorem is established and used to prove a conjecture of Crainic and Moerdijk on deformations of Lie brackets.

Keywords: Lie brackets; Homotopy; Differentiating; Deformations

1. Introduction

Some of the usual constructions of representations of Lie groups and Lie algebras can be extended to the case of Lie groupoids and algebroids only in the context of representations up to homotopy. These are respectively A_∞ and L_∞ versions of usual representations. For Lie groupoids, the adjoint representation is a representation up to homotopy which has the formal properties one would expect, for instance, with respect to the cohomology of the classifying space [2]. For Lie algebroids, the adjoint representation is a representation up to homotopy [1,6], whose associated cohomology controls the infinitesimal deformations of the structure, as expected from the case of Lie algebras.

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The purpose of this paper is to compare the global and the infinitesimal versions of representations up to homotopy, and to explain an application regarding the deformations of Lie brackets. We construct a differentiation functor $\Psi : \hat{\mathcal{R}}\mathrm{ep}^{\infty}(G) \to \mathcal{R}\mathrm{ep}^{\infty}(A)$ from the category of unital representations up to homotopy of a Lie groupoid to the category of representations up to homotopy of its Lie algebroid. Moreover, we show that given a representation up to homotopy E of G there is a homomorphism

$$\Psi : H(G, E) \to H(A, \Psi(E))$$

from the cohomology associated to E to the cohomology associated to $\Psi(E)$. We prove a van Est type theorem, which provides conditions under which this map is an isomorphism (in certain degrees).

In [5] Crainic and Moerdijk introduced a deformation complex for Lie algebroids. Its cohomology controls the deformations of the Lie algebroid structures and arises in the study of the stability of leaves of Lie algebroids [4]. Based on the rigidity properties of compact group actions, Crainic and Moerdijk stated a rigidity conjecture (Conjecture 1 of [5]) which gives conditions under which the deformation cohomology should vanish. In the case of a Lie algebra $\mathfrak{g}$, the conjecture corresponds to the vanishing of $H^2(\mathfrak{g}, \mathfrak{g})$, for $\mathfrak{g}$ semisimple of compact type. We will explain how representations up to homotopy can be used to give a proof of this conjecture. Since the deformation cohomology of a Lie algebroid coincides with the cohomology associated to the adjoint representation, one can reproduce the usual proof for Lie algebras [7–9], once the differentiation functor and the van Est isomorphism theorem have been established in the context of representations up to homotopy.

The paper is organized as follows. Section 2 contains the definitions and general facts about representations up to homotopy. In Section 3 we construct the differentiation functor for representations up to homotopy. The main result of this section is Theorem 3.14. We begin Section 4 by proving a van Est isomorphism theorem, see Theorem 4.7. We then show that by differentiating the adjoint representation of a Lie groupoid one obtains that of the Lie algebroid (Proposition 4.10). At the end we prove the rigidity conjecture of [5], which is Theorem 4.11.

2. Preliminaries

In order to fix our conventions, we will review the definitions and basic facts regarding representations up to homotopy of Lie algebroids and groupoids. More details on these constructions as well as the proofs of the results stated in this section can be found in [1,2]. Throughout the text, A will denote a Lie algebroid over a manifold M, and G will be a Lie groupoid over M.

Given a Lie algebroid A, there is a differential graded algebra $\Omega(A) = \Gamma(\Lambda A^*)$, with differential defined via the Koszul formula

$$\begin{aligned} \mathrm{d}\omega(\alpha_1, \ldots, \alpha_{n+1}) &= \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)} \omega(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1}), \end{aligned}$$

where ρ denotes the anchor map and $L_X(f) = X(f)$ is the Lie derivative along vector fields. The operator d is a coboundary operator ($d^2 = 0$) and satisfies the derivation rule

$$d(\omega\eta) = d(\omega)\eta + (-1)^p \omega d(\eta),$$

for all $\omega \in \Omega^p(A)$, $\eta \in \Omega^q(A)$.