Available online at www.sciencedirect.com

SciVerse ScienceDirect

Indagationes Mathematicae 24 (2013) 42–56

indagationes mathematicae

www.elsevier.com/locate/indag

On the metric theory of p-adic continued fractions

J. Hančl[a], A. Jaššová[b], P. Lertchoosakul[b], R. Nair[b,*]

[a] *Department of Mathematics and Centre for Excellence IT4Innovation, division of UO, Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, 30 dubna 22, 701 03 Ostrava 1, Czech Republic*

[b] *Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool L69 7ZL, United Kingdom*

Received 21 November 2011; received in revised form 3 June 2012; accepted 7 June 2012

Communicated by R. Tijdeman

Abstract

An analogue of the regular continued fraction expansion for the p-adic numbers for prime p was given by T. Schneider, such that for x in $p\mathbb{Z}_p$, i.e. the open unit ball in the p-adic numbers, we have uniquely determined sequences ($b_n \in \{1, 2, \ldots, p-1\}, a_n \in \mathbb{N}$) ($n = 1, 2, \ldots$) such that

$$x = \cfrac{p^{a_0}}{b_1 + \cfrac{p^{a_1}}{b_2 + \cfrac{p^{a_2}}{b_3 + \cfrac{p^{a_3}}{b_4 + \ddots}}}}.$$

A sample result that we prove is that if p_n ($n = 1, 2, \ldots$) denotes the sequences of rational primes, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{p_n}(x) = \frac{p}{p-1},$$

almost everywhere with respect to Haar measure. In the case where p_n is replaced by n this result is due to J. Hirsh and L. C. Washington. The proofs rely on pointwise subsequence and moving average ergodic theorems.

Keywords: Continued fraction; p-adic numbers

* Corresponding author. Tel.: +44 151 794 4057; fax: +44 151 794 4043.

E-mail addresses: hancl@osu.cz (J. Hančl), ajassova@liv.ac.uk (A. Jaššová), pojtitee@liv.ac.uk (P. Lertchoosakul), nair@liv.ac.uk (R. Nair).

doi:10.1016/j.indag.2012.06.004

1. Introduction

Extending the idea of the Euclidean algorithm, for a real number x, let

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 + \ddots}}}},$$

denote its *regular continued fraction expansion*, which is also written more compactly as $[c_0; c_1, c_2, \ldots]$. The terms $c_0, c_1, \ldots$ are called the *partial quotients* of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \ldots, c_n] = \frac{p_n}{q_n} \quad (n = 1, 2, \ldots),$$

are called the *convergents* of the continued fraction expansion.

For a real number y let $\{y\}$ denote its fractional part. We now consider the particular ergodic properties of the Gauss map, defined on [0, 1] by

$$Tx = \begin{cases} \left\{\dfrac{1}{x}\right\} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that $c_n(x) = c_{n-1}(Tx)$ $(n = 1, 2, \ldots)$. The dynamical system (X, β, μ, T) where X denotes [0, 1], β is the σ-algebra of Borel sets on X, $\mu = \gamma$ is the measure on (X, β) defined for any A in β by

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1},$$

and T is the Gauss map is ergodic. See [24], [8, pp. 165–177], or Chapter 4 of [10] for more details. This point of view can be used to prove results like the following.

Suppose $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous, increasing and such that

$$\int_0^1 |F(c_1(x))| dx < \infty.$$

For each $n \in \mathbb{N}$ and arbitrary real numbers $d_1, \ldots, d_n$ we set

$$M_{F,n}(d_1, \ldots, d_n) = F^{-1}\left[\frac{F(d_1) + \cdots + F(d_n)}{n}\right].$$

Then we have

$$\lim_{n\to\infty} M_{F,n}(c_1(x), \ldots, c_n(x)) = F^{-1}\left[\int_0^1 F(c_1(x)) dx\right],$$