Available online at www.sciencedirect.com

SciVerse ScienceDirect

Indagationes Mathematicae 24 (2013) 174–180

indagationes mathematicae

www.elsevier.com/locate/indag

On an identity by Chaundy and Bullard. II. More history

Tom H. Koornwinder[a,*], Michael J. Schlosser[b]

[a] Korteweg–de Vries Institute, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands
[b] Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Vienna, Austria

Received 28 June 2012; accepted 16 August 2012

Communicated by F. Beukers

Abstract

An identity by Chaundy and Bullard writes $1/(1-x)^n$ $(n=1,2,\ldots)$ as a sum of two truncated binomial series. In a paper which appeared in 2008 in Indag. Math. the authors surveyed many aspects of this identity. In the present paper we discuss much earlier occurrences of this identity in works by Hering (1868), de Moivre (1738) and de Montmort (1713). A relationship with Krawtchouk polynomials in work by Greville (1966) is also discussed.

Keywords: Negative binomial sum; Gauss hypergeometric series; Old games of chance; Smoothing filter; Krawtchouk polynomials

1. Introduction

In our paper [16] we surveyed the history of the often rediscovered formula

$$1=(1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k+x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k. \tag{1}$$

We attributed the formula to Chaundy and Bullard [2, p. 256] (1960). However, we later learnt that some giant steps back in time can be made to much earlier occurrences of this formula. Almost one century before Chaundy and Bullard the formula was given by Hering [14](1868). Then, with a jump of more than one century, the formula was found in the work of de Moivre [6]

* Corresponding author.
E-mail addresses: T.H.Koornwinder@uva.nl (T.H. Koornwinder), michael.schlosser@univie.ac.at (M.J. Schlosser).

0019-3577/$ - see front matter
doi:10.1016/j.indag.2012.08.003

(1738). Even 25 years earlier the formula was given in implicit form already by de Montmort [8] (1713).

The paper successively discusses these three early occurrences of the formulas. Next a correspondence between Samuel Pepys and Isaac Newton, having some relation with identity (1), is briefly discussed. We conclude with a much more recent connection with Krawtchouk polynomials which is implicit in a paper by Hering [11] from 1966.

2. Hering (1868)

In 1868 Hering [14, p. 14, formula 1)] derived:

$$(1-x)_n^{-m} = (1-x)^{-m} - (1-x)^{-m}\, x^n\, (1-\overline{1-x}\,)_m^{-n}. \tag{2}$$

Here $(1-x)_n^{-m}$ is the power series in x of $(1-x)^{-m}$ truncated after the n-th term. Similarly, $(1-\overline{1-x}\,)_m^{-n}$ is the power series in $1-x$ of $(1-(1-x))^{-n}$ truncated after the m-th term. Thus Hering already had (1).

Hering's proof is different from any of the proofs given in (1). For generic non-integer m he writes for the left-hand side of (2):

$$\begin{aligned}(1-x)_n^{-m} &= \sum_{k=0}^{n-1} \frac{(m)_k}{k!}\, x^k = \frac{(m)_{n-1}}{(n-1)!}\, x^{n-1}\, {}_2F_1\left(\begin{matrix}-n+1, 1\\ -m-n+2\end{matrix}; x^{-1}\right)\\ &= \frac{(m)_{n-1}}{(n-1)!}\, \frac{x^n}{x-1}\, {}_2F_1\left(\begin{matrix}-m+1, 1\\ -m-n+2\end{matrix}; \frac{1}{1-x}\right),\end{aligned} \tag{3}$$

where inversion of the order of summation is used in the second equality and Pfaff's transformation formula in the third equality. If m tends to a positive integer, the last ${}_2F_1$ becomes

$$\sum_{k=0}^{m-1} \frac{(-m+1)_k}{(-m-n+2)_k}\, (1-x)^{-k} + \sum_{k=m+n-1}^{\infty} \frac{(-m-n+k+2)_{n-1}}{(-m-n+2)_{n-1}}\, (1-x)^{-k}. \tag{4}$$

(Here, although not emphasized by Hering, we should require for convergence that $|x-1| > 1$. This can later be relaxed in (2) by analytic continuation.) After multiplication of (4) by $\binom{m+n-2}{n-1} x^n/(x-1)$ we can rewrite the first term (by inversion of the order of summation) as

$$-(1-x)^{-m}\, x^n \sum_{k=0}^{m-1} \frac{(n)_k}{k!}\, (1-x)^k = -(1-x)^{-m}\, x^n\, (1-\overline{1-x}\,)_m^{-n},$$

and the second term as

$$(-1)^n (1-x)^{-m-n} x^n \sum_{k=0}^{\infty} \frac{(n)_k}{k!}\, (1-x)^{-k} = (1-x)^{-m}.$$

Thus by substitution in (3) Hering settled (2).

Formula (2) is just one of many formulas derived in [14]. Hering does not specially emphasize this particular result.

3. de Moivre (1738)

A much earlier reference was kindly communicated to us by Pieter de Jong and also mentioned in his manuscript [5]. In 1738 de Moivre [6, p. 196] (see also the 1754 edition [7, p. 224]) wrote: