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On the distance between products of consecutive Fibonacci numbers and powers of Fibonacci numbers

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Abstract

Here, we find a lower bound for $|F_n \cdots F_{n+k-1} - F_m^\ell|$ for positive integers $k,\ \ell,\ m$ and n in terms of $\max\{k, \ell, m, n\}$, where F_s is the sth Fibonacci number.

Keywords: Fibonacci numbers; Applications of linear forms in logarithms of algebraic numbers

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

In this paper, we study a lower bound on the quantity

$$|F_n \cdots F_{n+k-1} - F_m^\ell| \quad \text{where } k \geq 1,\ \ell \geq 1,\ n \geq 3,\ m \geq 3. \tag{1}$$

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Since $F_1 = F_2 = 1$, our results give a lower bound for the expressions (1) for all quadruples of nonnegative integers (k, ℓ, m, n) in terms of $\max\{k, \ell, m, n\}$ except for some trivial cases such as when $n = 0$, or when $m \in \{0, 1, 2\}$, or when $m = n$ and $\ell = 1$.

Theorem 1. *Let (k, ℓ, m, n) be integers with the properties that $k \geq 1, \ell \geq 1, n \geq 3$ and $m \geq 3$. Put $X := \max\{k, \ell, m, n\}$. Then the inequality*

$$|F_n \cdots F_{n+k-1} - F_m^{\ell}| > 10^{-3/2} X^{1/40} \quad \textit{holds always} \tag{2}$$

except when either $\ell = k = 1$ and $m = n (= X)$ or $(k, \ell, m, n) = (1, 3, 3, 6)$, for which the left-hand side of Eq. (2) above is 0.

We have the following numerical corollary.

Corollary 1. *The largest solution of the inequality*

$$|F_n \cdots F_{n+k-1} - F_m^{\ell}| \leq 100 \quad \textit{with } k \geq 1,\ \ell \geq 1,\ m \geq 3,\ n \geq 3 \tag{3}$$

and $m \neq n$ when $k = \ell = 1$ is

$$|F_9 \cdots F_{13} - F_{11}^5| = 89. \tag{4}$$

Here, by the largest solution we mean the solution with the maximal value of $\max\{F_n \cdots F_{n+k-1}, F_m^{\ell}\}$ among all the possible solutions. This maximal value equals 5 584 059 449.

We note that formula (4) is a consequence of the formula

$$F_n F_{n+1} F_{n+3} F_{n+4} - F_{n+2}^4 = -1,$$

which holds for all positive integers n (in the particular case of (4) we have $n = 9$).

Throughout the paper, $(L_n)_{n \geq 0}$ denotes the Lucas companion of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. We write $(\alpha, \beta) := \left(\left(1 + \sqrt{5}\right)/2, (1 - \sqrt{5})/2\right)$ for the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci and Lucas sequences. The Binet formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 0. \tag{5}$$

We shall frequently use, often without mentioning it, the inequality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{for all positive integers } n.$$

2. The proof of Theorem 1

We fix an integer a and study the equation

$$F_n F_{n+1} \cdots F_{n+k-1} - F_m^{\ell} = a. \tag{6}$$

The plan is to show that it has only finitely many integer solutions with the properties $k \geq 1$, $\ell \geq 1$, $m \geq 3$ and $n \geq 3$, except when $a = 0$ and either $k = \ell = 1$ and $m = n$, or $(k, \ell, m, n) = (1, 3, 3, 6)$, and to bound X in terms of a. Note that if in (6) we replace products of