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Discrete-time path distributions on a Hilbert space

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This article is dedicated to the memory of Erik Thomas

Abstract

We construct a path distribution representing the kinetic part of the Feynman path integral at discrete times similar to that defined by Thomas (2000) [15], but on a Hilbert space of paths rather than a nuclear sequence space. We also consider different boundary conditions and show that the discrete-time Feynman path integral is well-defined for suitably smooth potentials.

Keywords: Feynman path integral; Distribution theory; Hilbert space of sequences; Wave operator

1. Motivation and basic set-up

1.1. Feynman path integral as a path distribution

In the Lagrangian formulation of quantum mechanics one defines the *action* of a particle as an integral of the Lagrangian over the time duration of the motion:

$$S(x_f, t_f; x_i, t_i) = \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t), t).$$

In general, the Lagrangian $L(x(t), \dot{x}(t), t)$ depends explicitly on the time, as well as on the position $x(t)$ and the velocity $\dot{x}(t)$ of the particle. For one-dimensional motion, the Lagrangian

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has the form

$$L(x(t), \dot{x}(t), t) = \frac{m}{2}\dot{x}(t)^2 - V(x(t), t),$$

where the first term is the kinetic energy term and $V(x(t), t)$ is the external potential. The time-evolution of a wave function $\Psi(x, t)$ is then given by

$$\Psi(x_f, t_f) = \int K(x_f, t_f; x_i, t_i)\ \Psi(x_i, t_i) dx_i, \tag{1.1}$$

where the *propagator* $K(x_f, t_f; x_i, t_i)$ is given by a *path integral* of the form

$$K(x_f, t_f; x_i, t_i) = \int e^{iS(x_f, t_f; x_i, t_i)/\hbar}\, \mathcal{D}[x(t)]. \tag{1.2}$$

Here $\mathcal{D}[x(t)]$ indicates a putative "continuous product" of Lebesgue measures $\mathcal{D}[x(t)] = \prod_{t\in(t_i,t_f)} dx(t)$. (Note that the action S above is a functional of the path $x(t)$.) It is a formidable mathematical challenge to make sense of this path-integral concept. Feynman himself interpreted it loosely as a limit of multidimensional integrals. However, as Thomas [15] remarks, even the finite-dimensional integrals are not proper integrals, though they can be defined as improper integrals. It was already noted by Cameron [6] that the path integral cannot be interpreted as a complex-valued measure. In fact, as Thomas [15] and Bijma [4] show, it cannot even be interpreted as a summable distribution because the summability order diverges as the number of integrals tends to infinity.

Various alternative approaches have been proposed to interpret the Feynman path integral as a limit of regularised integrals, e.g. [7,17,12]. The 'Euclidean approach' of 'Wick rotating' the time in the complex plane has led to the development of Euclidean quantum field theory, which has been the most successful way of constructing examples of quantum field theories. However, this still leaves open the question as to how the path integral object should be interpreted mathematically. De Witt-Morette [8] has argued that it should be a kind of distribution, but her approach was formal rather than constructive. The Itô–Albeverio–Høegh Krohn [1] approach was more constructive. They gave a definition of the path integral as a map from the space of Fourier transforms of bounded measures to itself and were able to show, using a perturbation expansion, that this is well-defined for potentials which are also Fourier transforms of bounded measures. Albeverio and Mazzucchi [2] later extended this approach to encompass polynomially growing potentials. This approach gives a mathematical meaning to the path-integral expression for the solution of the Schrödinger equation with initial wave function, rather than the propagator. A different approach in terms of functionals of white noise was proposed by Hida et al. [11]. In both approaches, the space of 'paths' is rather abstract.

In [15], Thomas initiated a different approach, with the aim of defining the path integral as a generalised type of distribution, in the spirit of De Witt-Morette, which he called a *path distribution*. In fact, this project is only at the beginning stages. In [15], he constructed an analogue of the path integral in discrete time, where the paths are sequences in a certain nuclear sequence space. His main idea is to define the path integral as a derivative of a measure, which we call the Feynman–Thomas measure. In this paper, we simplify his approach by defining the path distribution on a space of paths in a Hilbert space instead. This makes the construction more explicit and the technical details less demanding.

In the following, we set $m = 1$ and $\hbar = 1$ for simplicity. Discretising the action to a finite subdivision $\sigma = \{t_1, \dots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n < T$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we