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Symplectic rigidity and flexibility of ellipsoids☆

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Abstract

Rigidity is proved for symplectic embeddings of an ellipsoid into another of the same shape type, and new flexibility results are derived from a variant of the symplectic folding process.

Keywords: Symplectic topology; Embedding; Rigidity; Flexibility; Ellipsoids

A volume form on a smooth n-dimensional manifold M is a nowhere vanishing n-form Ω. On every open set $U \subset \mathbb{R}^n$ we consider the standard volume $\Omega_0 = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be volume preserving if:

$$\varphi^* \Omega = \Omega_0.$$

A *symplectic manifold* is a pair (M, ω), where M is a $2n$-dimensional differentiable manifold and ω is a symplectic form: a closed nondegenerate 2-form. Then:

$$\Omega = \frac{1}{n!}\,\omega^n \quad \text{is a volume form, and } \mathrm{d}\omega = 0.$$

A *symplectic map* is a map $\varphi : (M, \omega) \longrightarrow (M', \omega')$, such that:

$$\varphi^* \omega' = \omega.$$

Let $\mathcal{D}(n)$ be the group of symplectic diffeomorphisms, or symplectomorphisms, or canonical transformations, of $\mathbb{R}^{2n}$, and $\mathrm{Sp}(n)$ its subgroup of linear isomorphisms.

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On every open set $U \subset \mathbb{R}^{2n}$ we consider the standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be symplectic if it is a symplectic map:

$$\varphi^*\omega = \omega_0, \quad \text{and therefore } \varphi^*\Omega = \Omega_0$$

where Ω and Ω_0 are the volume forms induced by the symplectic forms.

Definition 1. An open symplectic ellipsoid of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radii $r_i = \sqrt{a_i/\pi}$ is the set:

$$E(a) = E(a_1, \ldots, a_n) = \left\{ z \;\middle|\; \frac{\pi |z_1|^2}{a_1} + \cdots + \frac{\pi |z_n|^2}{a_n} < 1 \right\},$$

where we assume $a_1 \leq \cdots \leq a_n$, and $z_j = x_j + i y_j$.

Definition 2. An open symplectic cylinder of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radius $r = \sqrt{a/\pi}$ is the set:

$$\begin{aligned} Z(a) &= \{(x, y) \in \mathbb{R}^{2n} : \pi |(x_1, y_1)|^2 < a\} \\ &= \{z \in \mathbb{C}^n : \pi |z_1|^2 < a\}. \end{aligned}$$

Remark 1. The ball of radius r is denoted by $B(\pi r^2)$:

$$B(a) = E(a, a, \ldots, a), \quad Z(a) = E(a, \infty, \ldots, \infty).$$

In dimension 2, an embedding is symplectic if and only if it is area and orientation preserving. In higher dimensions there exists symplectic rigidity, as first shown in [5]:

Gromov Theorem. *If there is a symplectic embedding $\varphi : B(a) \longrightarrow Z(A)$ of a ball into a symplectic cylinder, then $a \leq A$.*

The detection of embedding obstructions and the proof of the corresponding rigidity results will be based on symplectic capacities:

Definition 3. An *extrinsic symplectic capacity* c on $(\mathbb{R}^{2n}, \omega_0)$ is a map c such that, for every $A \subset \mathbb{R}^{2n}$, $c(A) \in [0, +\infty]$, satisfying the following properties:

Monotonicity: $c(A) \leq c(A')$ if there exists $\varphi \in \mathcal{D}(n)$ such that $\varphi(A) \subset A'$.
Conformality: $c(\alpha A) = \alpha^2 c(A)$, for any nonzero $\alpha \in \mathbb{R}$.
Nontriviality: $0 < c(B(\pi)), c(Z(\pi)) < \infty$.

The flexibly results we present are based on an explicit construction of the symplectic embedding, and try to put in evidence the importance of the shape of the ellipsoids involved. The recent results on optimal embeddings in [11–13] are based in non-explicit methods specific of complex dimension $n = 2$ as they do not have a straightforward generalization to $n > 2$; the results were extended to higher dimensions in [1].

1. Rigidity

When considering linear symplectic embeddings, there exists symplectic rigidity: