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Operator measures and integration operators

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Abstract

Let (Ω, Σ, μ) be a finite complete measure space and $(X, \|\cdot\|_X)$ be a Banach space with the Banach dual X^*. Let $\mathcal{L}^\infty(\mu, X)$ denote the space of all μ-measurable functions $f : \Omega \to X$ such that $\operatorname{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X < \infty$. We study the problem of the integral representation of some natural classes of linear operators from $\mathcal{L}^\infty(\mu, X)$ to a Banach space with respect to the corresponding operator measures. We characterize relatively $\sigma(\mathrm{bvca}_\mu(\Sigma, X^*), \mathcal{L}^\infty(\mu, X))$-sequentially compact sets in the space $\mathrm{bvca}_\mu(\Sigma, X^*)$ of all countably additive measures $\nu : \Sigma \to X^*$ of bounded variation with $\nu(A) = 0$ if $\mu(A) = 0$.

Keywords: Operator measures; Strongly measurable functions; Weak-star sequential compactness; Absolutely continuous operators; Integration operators; σ-smooth operators

1. Introduction and terminology

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and let B_X stand for the closed unit ball in X. Let X^* and Y^* stand for the Banach duals of X and Y respectively. Denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. By $\sigma(L, K)$ we will denote the weak topology with respect to a dual pair $\langle L, K\rangle$.

Now we recall basic terminology concerning operator measures (see [9–12,2,15,16,21]). Let Σ be a σ-algebra of subsets of a non-empty set Ω. For a finitely additive measure $m : \Sigma \to \mathcal{L}(X, Y)$ we define the semivariation $\widetilde{m}$ on $A \in \Sigma$ by $\widetilde{m}(A) := \sup \|\Sigma m(A_i)(x_i)\|_Y$, where the supremum is taken over all finite disjoint sequences (A_i) in Σ with $A_i \subset A$ and $x_i \in B_X$ for each i.

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By $\operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ we denote the set of all finitely additive measures $m : \Sigma \to \mathcal{L}(X, Y)$ with finite semivariation, i.e., $\widetilde{m}(\Omega) < \infty$.

For $y^* \in Y^*$ let $m_{y^*} : \Sigma \to X^*$ be a measure defined $m_{y^*}(A)(x) := \langle m(A)(x), y^* \rangle$ for $A \in \Sigma$ and $x \in X$. Moreover, for $A \in \Sigma$ we have (see [2, Theorem 5]):

$$\widetilde{m}(A) = \sup\{|m_{y^*}|(A) : y^* \in B_{Y^*}\},$$

where $|m_{y^*}|(A)$ stands for the variation of m_{y^*} on $A \in \Sigma$ (see [2, Theorem 5]). A measure $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is said to be *variationally semi-regular* if $\widetilde{m}(A_n) \to 0$ whenever $A_n \downarrow \emptyset$ and $(A_n) \subset \Sigma$, i.e., the family $\{|m_{y^*}| : y^* \in B_{Y^*}\}$ is uniformly countably additive (see [15,16]). (Dobrakov [12] uses the term "continuous", Swartz [21] uses the term "strongly bounded").

By $\mathcal{S}(\Sigma, X)$ we denote the set of all X-valued Σ-simple functions $s = \sum_{i=1}^{k} (\mathbb{1}_{A_i} \otimes x_i)$, where $(A_i)_{i=1}^{k}$ is a disjoint sequence in Σ, $x_i \in X$ for $1 \le i \le n$ and $(\mathbb{1}_{A_i} \otimes x_i)(\omega) = \mathbb{1}_{A_i}(\omega) x_i$ for $\omega \in \Omega$. A function $f : \Omega \to X$ is said to be *strongly Σ-measurable* if there exists a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ for all $\omega \in \Omega$. It is known that if $f : \Omega \to X$ is strongly Σ-measurable, then there exists a sequence (s_n) in $\mathcal{S}(\Sigma, X)$ such that $\|s_n(\omega) - f(\omega)\|_X \to 0$ for all $\omega \in \Omega$ and $\|s_n(\omega)\|_X \le \|f(\omega)\|_X$ for $\omega \in \Omega$ and all $n \in \mathbb{N}$ (see [10, Theorem 1.6, p. 4]). By $\mathcal{L}^\infty(\Sigma, X)$ we denote the Banach space of all bounded strongly Σ-measurable functions $f : \Omega \to X$, equipped with the supremum norm $\|\cdot\|$.

Following [18, Definition 1.1] one can distinguish an important class of operators from $\mathcal{L}^\infty(\Sigma, X)$ to Y.

Definition 1.1. A bounded linear operator $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is said to be *σ-smooth* if $\|T(f_n)\|_Y \to 0$ whenever $\|f_n(\omega)\|_X \to 0$ for all $\omega \in \Omega$ and $\sup_n \|f_n\| < \infty$.

Assume that $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular. Then each function $f \in \mathcal{L}^\infty(\Sigma, X)$ is m-integrable and the integral of f on a set $A \in \Sigma$ is defined by the equality:

$$\int_A f\, dm := \lim \int_A s_n\, dm,$$

where (s_n) is a sequence in $\mathcal{S}(\Sigma, X)$ such that $\|(\mathbb{1}_A s_n)(\omega) - (\mathbb{1}_A f)(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $\sup_n \|\mathbb{1}_A s_n\| \le \|\mathbb{1}_A f\|$ (see [12, Definition 2, p. 523 and Theorem 5, p. 524]). In [18] we study the integration operator $T_m : \mathcal{L}^\infty(\Sigma, X) \to Y$ defined by $T_m(f) = \int_\Omega f\, dm$.

Proposition 1.1 (*See [18, Proposition 2.2]*). *Assume that $m \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular. Then the integration operator $T_m : \mathcal{L}^\infty(\Sigma, X) \to Y$ is σ-smooth.*

For a bounded linear operator $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ let $m_T : \Sigma \to \mathcal{L}(X, Y)$ stand for its *representing measure*, i.e.,

$$m_T(A)(x) := T(\mathbb{1}_A \otimes x) \quad \text{for } A \in \Sigma \text{ and } x \in X.$$

Then $\widetilde{m}_T(\Omega) \le \|T\| < \infty$, i.e., $m_T \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$.

Proposition 1.2 (*See [18, Proposition 2.3]*). *Assume that $T : \mathcal{L}^\infty(\Sigma, X) \to Y$ is a σ-smooth operator. Then its representing measure $m_T \in \operatorname{fasv}(\Sigma, \mathcal{L}(X, Y))$ is variationally semi-regular and*

$$T(f) = T_{m_T}(f) = \int_\Omega f\, dm_T \quad \text{for all } f \in \mathcal{L}^\infty(\Sigma, X).$$

Moreover, $\|T\| = \widetilde{m}_T(\Omega)$.