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Stieltjes interlacing of zeros of Laguerre polynomials from different sequences

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Abstract

Stieltjes' Theorem (cf. Szegö (1959) [10]) proves that if $\{p_n\}_{n=0}^{\infty}$ is an orthogonal sequence, then between any two consecutive zeros of p_k there is at least one zero of p_n for all positive integers k, $k < n$; a property called Stieltjes interlacing. We prove that Stieltjes interlacing extends across different sequences of Laguerre polynomials L_n^{α}, $\alpha > -1$. In particular, we show that Stieltjes interlacing holds between the zeros of $L_{n-1}^{\alpha+t}$ and L_{n+1}^{α}, $\alpha > -1$, when $t \in \{1, \ldots, 4\}$ but not in general when $t > 4$ or $t < 0$ and provide numerical examples to illustrate the breakdown of interlacing. We conjecture that Stieltjes interlacing holds between the zeros of $L_{n-1}^{\alpha+t}$ and those of L_{n+1}^{α} for $0 < t < 4$. More generally, we show that Stieltjes interlacing occurs between the zeros of L_{n+1}^{α} and the zeros of the kth derivative of L_n^{α}, as well as with the zeros of $L_{n-k}^{\alpha+k+t}$ for $t \in \{1, 2\}$ and $k \in \{1, 2, \ldots, n-1\}$. In each case, we identify associated polynomials, analogous to the de Boor–Saff polynomials (cf. de Boor and Saff (1986) [3], Ismail (2005) [6]), that are completely determined by the coefficients in a mixed three-term recurrence relation, whose zeros complete the interlacing process.

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1. Introduction

A classical theorem of Stieltjes (cf. [10, Theorem 3.3.3]) proves that if $\{p_n\}_{n=0}^{\infty}$ is any sequence of orthogonal polynomials then the zeros of p_k and p_n, $k < n$, are interlacing in the sense that each open interval of the form $(-\infty, z_1)$, $(z_1, z_2), \ldots, (z_{k-1}, z_k)$, (z_k, ∞), where $z_1 < z_2 < \cdots < z_k$ are the zeros of p_k, contains at least one zero of p_n. If we assume that p_k and p_n have no common zeros, the same argument used by Stieltjes shows that there exist k open intervals, with endpoints at successive zeros of p_n, each of which contains exactly one zero of p_k.

For polynomials p_k and p_n with $k < n-1$, there are clearly not enough zeros of p_k to provide the required number of points to interlace fully with the zeros of p_n. However, this deficit in the number of points needed to complete the interlacing process between the zeros of polynomials of non-consecutive degree in an orthogonal sequence is well understood. In [2, Theorem 4], Beardon proves that if $\{p_n\}_{n=0}^{\infty}$ is any sequence of orthogonal polynomials with p_m and p_n having no common zeros for $m \neq n$, then there exist real polynomials S_{n-m-1} of degree $n-m-1$ whose real simple zeros, together with the zeros of p_m, interlace with the zeros of p_n for $m < n$. The polynomials S_{n-m-1} were first observed, albeit in a rather different context, by de Boor and Saff in [3] and are also studied by Vinet and Zhedanov in [11]. An important feature of the polynomials S_{n-m-1} is that they are completely determined by the coefficients in the three-term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ (cf. [2]). Independently, it also follows immediately from Segura's result (cf [8], Theorem 1) that the zero of the linear polynomial that completes the interlacing of the zeros of p_{n-1} with those of p_{n+1} is given by one of the coefficients in the three-term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$.

The question arises as to whether Stieltjes interlacing occurs between the zeros of two polynomials p_n and q_k, $k < n-1$, from different orthogonal sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ and whether polynomials analogous to the de Boor–Saff polynomials exist in this more general situation. Among the classical orthogonal families of Gegenbauer, Laguerre and Jacobi polynomials, natural choices of different orthogonal sequences are those corresponding to different values of the appropriate parameter(s) and also the (orthogonal) sequences of their derivatives.

In this paper we prove that Stieltjes interlacing takes place between the zeros of Laguerre polynomials L_{n+1}^{α} and $L_{n-1}^{\alpha+t}$, $n \in \mathbb{N}$, $\alpha > -1$, $t \in \{1, 2, 3, 4\}$ and we identify the polynomials that are analogous to the de Boor–Saff polynomials in these cases. We make two conjectures regarding the location of the (single) extra point that completes the interlacing process for continuous variation of the parameter t in the range $0 < t < 2$ and $2 < t < 4$. Our two conjectures are equivalent when $t = 2$. Numerical examples are provided to illustrate that Stieltjes interlacing breaks down in general when $t = 5$ or $t = -1$. Our main result proves that Stieltjes interlacing occurs between the zeros of L_{n+1}^{α} and the zeros of the kth derivative of L_n^{α} for all $k \in \{1, \ldots, n-1\}$ and identifies the polynomials analogous to the de Boor–Saff polynomials. Finally, we prove that Stieltjes interlacing holds between the zeros of L_{n+1}^{α} and the zeros of $L_{n-k}^{\alpha+k+t}$ for $t = 1$ and $t = 2$.

We recall that, for $\alpha > -1$, the Laguerre polynomials L_n^{α} are orthogonal with respect to the weight function $w_{\alpha}(x) = x^{\alpha} e^{-x}$ on $(0, \infty)$ and satisfy the three-term recurrence relation

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+\alpha+1-x)L_n^{\alpha}(x) - (\alpha+n)L_{n-1}^{\alpha}(x). \tag{1}$$

In Section 2, we discuss the existence of common zeros of Laguerre polynomials. Thereafter, we will assume in all our theorems, lemmas and conjectures that the polynomials under