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Algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$

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ABSTRACT

In this note we classify, up to conjugation, all algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$.

1. INTRODUCTION

Although the classification, up to conjugation, of the algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$ ([3, Theorem 4.12], [6, Theorem 4.29]), and the classification of subgroups of GL_2 over a finite field ([1], [8, Theorem 6.17]) are well known, it seems that the determination of all algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$ is not presented well in the literature. In this paper we give this classification, including full proofs. The final result is Theorem 4. We note that $\mathbb{C}$ can be replaced everywhere by any algebraically closed field of characteristic zero.

Notation. $\mu_n \subset \mathbb{C}^*$ denotes the nth roots of unity and ζ_n denotes a primitive nth root of unity. Let $\beta : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$, $\gamma : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ denote the canonical projections. For any algebraic subgroup $H \subset \mathrm{PSL}_2(\mathbb{C})$ we write $H^{\mathrm{SL}_2} = \gamma^{-1}(H) \subset \mathrm{SL}_2(\mathbb{C})$. Further

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

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and

$$D_\infty := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \,\middle|\, c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} \,\middle|\, d \in \mathbb{C}^* \right\}$$

are the Borel subgroup and the infinite dihedral subgroup of $\mathrm{SL}_2(\mathbb{C})$.

We first recall the classification of all algebraic subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

Theorem 1. *Let H be an algebraic subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Then, up to conjugation, one of the following cases occurs:*

(1) *$H = \mathrm{PGL}_2(\mathbb{C})$;*
(2) *H is a subgroup of the group $\gamma(B)$;*
(3) *$H = \gamma(D_\infty)$;*
(4) *$H = D_n$ (the dihedral group of order $2n$), A_4 (the tetrahedral group), S_4 (the octahedral group), or A_5 (the icosahedral group).*

The above theorem reduces the problem to describing the algebraic groups in $\mathrm{GL}_2(\mathbb{C})$ mapping to a given subgroup $G \subset \mathrm{PGL}_2(\mathbb{C})$. Each example is therefore a central extension of G and corresponds to an element in $H^2(G, \mu)$, where μ is either $\mathbb{C}^*$ or a finite cyclic subgroup of $\mathbb{C}^*$. The first case defines the Schur multiplier of G. In the interesting cases, μ is a finite group and the Schur multiplier does not provide information because the canonical map $H^2(G, \mu) \to H^2(G, \mathbb{C}^*)$ is not injective (see also Remark 3).

We note that Theorem 1 is a corollary of the following two well-known theorems.

Theorem 2 (Klein [4]). *A finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to one of the following polyhedral groups:*

- *a cyclic group C_n;*
- *a dihedral group D_n of order $2n$, $n \geqslant 2$;*
- *the tetrahedral group A_4 of order 12;*
- *the octahedral group S_4 of order 24;*
- *the icosahedral group A_5 of order 60.*

Up to conjugation, all of these groups occur as subgroups of $\mathrm{PGL}_2(\mathbb{C})$ exactly once.

In Theorem 1, the cyclic groups C_n happen to be subgroups of $\gamma(B)$.

Theorem 3 ([3, Theorem 4.12]; [6, Theorem 4.29]). *Suppose that G is an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then, up to conjugation, one of the following cases occurs:*

(1) *$G = \mathrm{SL}_2(\mathbb{C})$;*
(2) *G is a subgroup of the Borel group B;*