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(r, p)-capacity and Hausdorff measure on a local field

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ABSTRACT

We will see a relationship between capacities and Hausdorff measure on local field. We can rely on technique in the existing capacity theory on the Euclidean space, however some counterparts of fundamental facts such as Whitney decomposition theorem and Wolff's inequality will be required. After establishing those counterparts, we will see some potential theoretic features based on the stable process on the field.

1. INTRODUCTION

Stochastic processes on the field Q_p of p-adic numbers have been studied in many papers such as Albeverio and Karwowski [2,3], Albeverio and Zhao [8–10], Evans [14], Figà-Talamanca et al. [12,17], Kaneko [27–29], Karwowski and Vilela Mendes [32], Kochubei [36–38], Yasuda [42–44], Vladimirov et al. [40,41], etc. In some of these articles, stochastic process on p-adics is called random walk on p-adics.

Various structural grips on stochastic processes on p-adics with real time parameter have been also made in those researches. More specifically, Albeverio and Karwowski gave intuitively understandable perspectives of stochastic processes on p-adics in [2,3] including the description with Dirichlet spaces of the stochastic processes. Karwowski and Vilela Mendes [32] found larger families by a similar approach based on the simple and clear structural perspectives. In [43], Yasuda started with random sums of locally compact abelian group valued random variables

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and discussed the convergence of the measures induced on the path space so that stochastic processes in [2,3] are obtained as the limits. In [27], the author imported a language of Dirichlet spaces to introduce a family of stochastic processes and demonstrated comparison principle which is taken for granted in analysis when two comparable infinitesimal generators are given. Some potential theoretic features such as Harnack inequalities, equilibrium measure and α-capacity based on Riesz potential of α-stable process were explored by Haran [20].

Recently, in [38], Kochubei shown some features of Hausdorff measure on an infinite extension of $\boldsymbol{Q}_p$. Yasuda and the author found some properties of non-linear capacity on the extension in [30], after some preliminary observation on non-linear capacity on finite extensions over $\boldsymbol{Q}_p$, where a counterpart of Sobolev space introduced by Fukushima and the author [18] played a pivotal role. The counterpart is associated fundamentally with a Bessel kernel and it has been investigated by many researchers. For instance, we can look at recent development of the theory in the articles by Farkas, Hirsh, Hoh, Jacob, Kazumi, Schilling, Shigekawa and Song [15,16,21–24,33]. In those articles, we see many aspects of this theory which are tightly related to the non-linear capacity theory associated with the ordinary Sobolev space.

In Section 2, we will start with a nice subclass of stochastic processes initiated by Albeverio and Karwowski and closely look at the non-linear capacity based on Fukushima and the author's paper [18,26], aided by the theory reconciled in [1] based on ordinary analysis. In Section 3, some basic facts such as Hardy–Littlewood–Wiener theorem and Whitney decomposition will be shown on a local field. We will derive some other basic inequalities where a counterpart $\mathcal{V}_r^{(\alpha)}$ of Bessel kernel defined from the α-stable process is involved. In Section 4, we will see that Wolff's inequality is valid on a local field.

In Section 5, we will notice that if two extensions K', K of $\boldsymbol{Q}_p$ with $K' \subset K$ are given, then K contains K' as a d-set with respect to the Haar measure $\mu_{K'}$ on K':

Theorem 2. *If K and K' are finite extensions of $\boldsymbol{Q}_p$ satisfying $K' \subset K$, then the restriction $H^{(\infty)}_{K',m_{K'}}$ of the Hausdorff content determined by the function $g(x) = x^{m_{K'}}$ to the set K' satisfies the following properties*:

(i) $H^{(\infty)}_{K',m_{K'}}(E) = \mu_{K'}(E \cap K')$, *for any measurable subset E of K*,
(ii) *there exists some positive constant $c_{K,K'}$ such that*

$$c_{K,K'} q_K^{km_{K'}/m_K} \leqslant H^{(\infty)}_{K',m_{K'}}\big(B\big(x, q_K^{k/m_K}\big)\big) \leqslant q_K^{km_{K'}/m_K}$$

for any $x \in K$ and integer k.

Here and in the sequel, $B(x, q_K^{k/m_K})$ stands for the ball in K centered at x with radius q_K^{k/m_K}. As for the definition of d-set, the reader can consult the monograph [25].

Finally, in the section we will find some inequalities revealing a relationship between non-linear capacities and Hausdorff content on finite extension of $\boldsymbol{Q}_p$, which can be viewed as counterpart of the results obtained in the Euclidean space.