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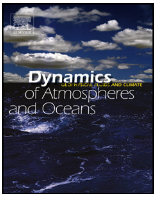

Revisiting well-posed boundary conditions for the shallow water equations

Sarmad Ghader[a,*], Jan Nordström[b]

[a] *Institute of Geophysics, University of Tehran, Tehran, Iran*
[b] *Division of Computational Mathematics, Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden*

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ABSTRACT

We derive a general form of well-posed open boundary conditions for the two-dimensional shallow water equations by using the energy method. Both the number and the type of boundary conditions are presented for subcritical and supercritical flows on a general domain. The boundary conditions are also discussed for a rectangular domain. We compare the results with a number of often used open boundary conditions and show that they are a subset of the derived general form.

1. Introduction

The single layer shallow water models are extensively used in numerical studies of large scale atmospheric and oceanic motions. This model describes a fluid layer of constant density in which the horizontal scale of the flow is much greater than the layer depth. The dynamics of the single layer model is less general than three dimensional models, but is often preferred because of its mathematical and computational simplicity (Pedlosky, 1987; Vallis, 2006).

Well-posed boundary conditions are an essential requirement for all stable numerical schemes developed for initial boundary value problems. For any limited area atmospheric or oceanic numer-

* Corresponding author. Tel.: +98 2161118373.
E-mail addresses: sghader@ut.ac.ir (S. Ghader), jan.nordstrom@liu.se (J. Nordström).

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ical model the lateral boundaries of the domain are not physical boundaries and artificial boundary conditions are used. The boundary conditions used at these artificial boundaries must lead to a well-posed problem. In addition, the boundary conditions must generate stable solutions for the discretized version of the governing equations. Furthermore, the boundaries should be as transparent as possible. The focus of the present work is on the first crucial step in this chain, namely the derivation of a general form of the well-posed boundary conditions for the two-dimensional shallow water equations and the other issues are not considered here.

For the one-dimensional shallow water equations, well-posed boundary conditions have been derived previously by transforming them into a set of decoupled scalar equations (Durran, 2010). Oliger and Sundström (1978) derived well-posed boundary conditions for several sets of partial differential equations including the shallow water equations by using the energy method. We will follow the path set by Oliger and Sundström (1978) and derive a general form of well-posed open boundary conditions for the two-dimensional shallow water equations. The core mathematical tool is the energy method where one bounds the energy of the solution by choosing a minimal number of suitable boundary conditions (Gustafsson et al., 1995; Nordström and Svärd, 2005; Gustafsson, 2008).

Secondly, it is shown that the open boundary conditions proposed by others such as Oliger and Sundström (1978), Mcdonald (2002) and Blayo and Debreu (2005) are special cases of the new general form of derived open boundary conditions in this paper.

The remainder of this paper is organized as follows. The shallow water equations are given in Section 2. Section 3 gives the various definitions of well-posedness. In Sections 4 and 5, well-posedness of the shallow water equations and well-posed boundary conditions for the two dimensional shallow water equations for a general domain are derived. The boundary conditions for a rectangular domain, as an example, are also presented. The details of the proposed open boundary conditions and the relation to work of others are presented in Section 6. Finally, concluding remarks are given in Section 7.

2. The shallow water equations

The inviscid single-layer shallow water equations, including the Coriolis term, are (Vallis, 2006)

$$\frac{D\mathbf{V}}{Dt} + f\hat{\mathbf{k}} \times \mathbf{V} + g\nabla h = 0 \tag{1}$$

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{V} = 0 \tag{2}$$

where $\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}$ is the horizontal velocity vector with u and v being the velocity components in x and y directions, respectively. $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors in x and y directions, respectively. h represents the surface height, $D()/Dt = \partial()/\partial t + (\mathbf{v} \cdot \nabla)()$ is the substantial time derivative, f is the Coriolis parameter and g is the acceleration due to gravity. The unit vector in vertical direction is denoted by $\hat{\mathbf{k}}$. Here, we use the f-plane approximation where the Coriolis parameter is taken to be a constant.

The vector form of the two-dimensional shallow water equations, linearized around a constant basic state, can be written as

$$\mathbf{u}_t + A\mathbf{u}_x + B\mathbf{u}_y + C\mathbf{u} = 0 \tag{3}$$

where the subscripts t, x and y denote the derivatives. The definition of the vector $\mathbf{u}$ and the matrices A, B and C is

$$\mathbf{u} = \begin{pmatrix} u' \\ v' \\ h' \end{pmatrix}, \quad A = \begin{pmatrix} U & 0 & g \\ 0 & U & 0 \\ H & 0 & U \end{pmatrix}, \quad B = \begin{pmatrix} V & 0 & 0 \\ 0 & V & g \\ 0 & H & V \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, u' and v' are the perturbation velocity components and h' is the perturbation height. In addition, U, V and H represent the constant mean fluid velocity components and height.