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Analytical solution for the periodic temperature field in frozen ground with account for variation in mean daily air temperature

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ABSTRACT

This paper presents an analytical solution for temperature waves in a two-layer medium consisting of frozen ground and snow cover. Temperature waves represent diurnal temperature fluctuations relative to the mean daily value. In the literature they are known for a constant mean daily temperature value. In this paper, a solution is obtained for periodic temperature of the ground–snow system with account for variations in mean daily air temperature over the entire winter. The solution can be used in estimating the effect of snow cover on the thermal state of frozen ground in the cold season.

1. Introduction

The temperature field in soils resulting from the effect of climatic conditions is affected by surface covers (snow, vegetation, moss/peat, etc.) which have highly different thermal properties. Observed diurnal, annual and other periodic variations in ground temperature are commonly interpreted based on the solution of the heat conduction equation for a semi-infinite medium whose surface temperature varies periodically (sinusoidally), where the problem space is represented as a multi-layer system. The literature dealing with the theoretical analysis of the periodic temperature field in multi-layer mediums is not voluminous. The two-layer case was treated by Ogilvi (1959) and Shastkevich (1973). Lachenbruch (1959) investigated the three-layer medium. For a greater number of layers, the mathematical apparatus becomes very complicated. These studies present a traditional, classical approach to solving the heat conduction equation. Another approach to the given problem is the matrix method commonly used in electric circuit theory. For the half space consisting of an arbitrary number of layers, Shklover (1952) and Carslaw and Jaeger (1959) used the complex variable method. This method allows one to obtain numerical values of the temperature wave parameters for special cases only.

All the above studies assume that the mean periodic (diurnal, annual, etc.) temperature is constant. However, the mean daily temperature varies considerably through the winter. The aim of this paper is to account for variation of mean daily air temperature with time when predicting the periodic temperature field in frozen ground covered with snow in the winter. This will permit more precise evaluation of the effect of snow cover on the thermal state of permafrost, which is of important scientific value for geocryological research. This approach will also have practical engineering applications, for example, in the use of insulation to prevent deep cooling of frozen ground during open-cast mining in permafrost regions.

2. General statement of the problem

The problem considers the two-layer medium consisting of the snow cover and the frozen ground represented as the semi-infinite space. It is assumed that both mediums have initial temperatures in the form of a constant gradient distribution with depth, each with its own temperature gradient. For simplicity of mathematical operations, air temperature in the boundary condition for the two-layer medium can be well replaced with snow surface temperature. They are close to each other. Pavlov's (1975) data indicate that the difference between mean daily air and snow surface temperature averaged for the winter is no greater than 1.0 to 1.8 °C, with snow surface temperature always lower than air temperature at 2 m height.

A diurnal temperature fluctuation is applied to the snow surface as a regular sinusoid relative to the daily mean value. The mean daily temperature varies with time. This temperature variation can be expressed as a piecewise linear function of time.

The conditions of the problem are shown in Fig. 1. The subscripts 1 and 2 denote the covering layer (snow cover) and the semi-infinite space (frozen ground), respectively. The following notations are used in Fig. 1: x — spatial coordinate, τ — time, t — temperature, λ — thermal conductivity, a — thermal diffusivity, A_0 — amplitude of daily temperature variations, and ω — angular frequency of daily temperature variations equal to $2\pi/T$ (here T is the period of variations equal to 1 day). The snow surface is at plane $x = -l$, and the interface between the mediums is at $x = 0$.

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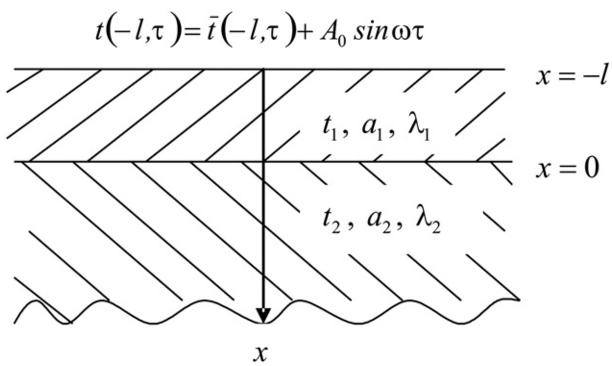

Fig. 1. Conditions of the problem for periodic temperature field in a two-layer medium.

The complete changes of temperature in the mediums considered $t_1(x, \tau)$ and $t_2(x, \tau)$ represent the daily temperature fluctuations $\tilde{t}_1(x,\tau)$ and $\tilde{t}_2(x,\tau)$ relative to the mean daily values $\bar{t}_1(x,\tau)$ and $\bar{t}_2(x,\tau)$. This can be expressed mathematically as

$$t_1(x,\tau) = \bar{t}_1(x,\tau) + \tilde{t}_1(x,\tau), -l<x<0 \quad (1)$$

$$t_2(x,\tau) = \bar{t}_2(x,\tau) + \tilde{t}_2(x,\tau), 0<x<\infty \quad (2)$$

3. Prediction of daily temperature fluctuation

In Eqs. (1) and (2), the daily temperature fluctuations can be calculated using Lachenbruch's equations for the two-layer medium, which for the steady periodic field take the form (Lachenbruch, 1959):

$$\tilde{t}_1(x,\tau) = A_0 e^{-x\sqrt{\frac{\omega}{2a_1}}}\frac{1}{S}\left[\sin\left(\omega\tau - x\sqrt{\frac{\omega}{2a_1}}\right) + Me^{2x\sqrt{\frac{\omega}{2a_1}}}\sin\left(\omega\tau + x\sqrt{\frac{\omega}{2a_1}}\right) + M\sin\left(\omega\tau - x\sqrt{\frac{\omega}{2a_1}}\right) + M^2 e^{2x\sqrt{\frac{\omega}{2a_1}}}\sin\left(\omega\tau + x\sqrt{\frac{\omega}{2a_1}}\right)\right] \quad (3)$$

at $-l<x<0$

$$\tilde{t}_2(x,\tau) = A_0(1+M)e^{-x\sqrt{\frac{\omega}{2a_1}}}\frac{1}{S}\left[\sin\left(\omega\tau - x\sqrt{\frac{\omega}{2a_1}}\right) + M\sin\left(\omega\tau - x\sqrt{\frac{\omega}{2a_1}}\right)\right] \quad (4)$$

at $0<x<\infty$, where $M = \frac{\beta_1-\beta_2}{\beta_1+\beta_2}$, $\beta = \sqrt{\lambda c_\gamma}$ is the thermal contact coefficient and c_γ is the volumetric heat capacity.

In Eqs. (3) and (4), the expression for S is

$$S = 1 + 2M + M^2. \quad (5)$$

4. Statement of the mean daily temperature problem

For the mean daily temperature, we formulate the following problem

$$a_1\frac{\partial^2 \bar{t}_1}{\partial x^2} = \frac{\partial \bar{t}_1}{\partial \tau}, -l < x < 0, \tau > 0 \quad (6)$$

$$a_2\frac{\partial^2 \bar{t}_2}{\partial x^2} = \frac{\partial \bar{t}_2}{\partial \tau}, 0 < x < \infty, \tau > 0. \quad (7)$$

The mean daily temperature at the surface of the snow ($x=-l$) is a variable which, in the general case, can be represented by a piecewise linear function of time:

$$\bar{t}_1(-l,\tau) = \bar{t}_1(-l,0) + b_1\tau_1 + b_2(\tau_2-\tau_1) + b_3(\tau_3-\tau_2) + \dots = \bar{t}_1(-l,0) + \sum_1^n b_i(\tau_i-\tau_{i-1}) \quad (8)$$

where $b_1, b_2, b_3 \dots b_i$ are the slopes of curves of snow surface temperature variation over the time intervals $\Delta\tau_i = \tau_i - \tau_{i-1}$ and n is the maximum number of time intervals.

The problem will be solved within each time interval for a linear variation in snow temperature:

$$\bar{t}_{1i}(-l,\Delta\tau_i) = \bar{t}_1(-l,\tau_{i-1}) + b_i\Delta\tau_i.$$

For simplicity of notation, the subscripts i and Δ in this expression will be omitted from here on

$$\bar{t}_1(-l,\tau) = \bar{t}_1(-l,0) + b\tau.$$

Let us consider the conditions of the problem.

Initial conditions:

$$\bar{t}_1(x,0) = \bar{t}_1(-l,0) + G_1(x+l) \quad (9)$$

$$\bar{t}_2(x,0) = \bar{t}_1(0,0) + G_2x = \bar{t}_1(-l,0) + G_1l + G_2x \quad (10)$$

where G_1 and G_2 are the initial temperature gradients in medium 1 and medium 2.

Boundary conditions:

at $x=-l$

$$\bar{t}_1(-l,\tau) = \bar{t}_1(-l,0) + b\tau \quad (11)$$

at $x=0, \tau>0$

$$\lambda_1\frac{\partial \bar{t}_1(0,\tau)}{\partial x} = \lambda_2\frac{\partial \bar{t}_2(0,\tau)}{\partial x} \quad (12)$$

$$\bar{t}_1(0,\tau) = \bar{t}_2(0,\tau) \quad (13)$$

at $x\to\infty$

$$\bar{t}_2(\infty,\tau) = \bar{t}_2(\infty,0). \quad (14)$$

5. Solution of the problem

Let us introduce the notations $\vartheta_1(x,\tau) = \bar{t}_1(x,\tau) - \bar{t}(x,0)$ and $\vartheta_2(x,\tau) = \bar{t}_2(x,\tau) - \bar{t}_2(x,0)$. This gives a problem with zero initial and boundary (at $x\to\infty$) conditions. All the other conditions and Eqs. (1) and (2) remain the same in the notations ϑ_1 and ϑ_2.

Apply the Laplace transforms to the system of Eqs. (1)–(6):

$$\bar{\vartheta}_1 = \int_0^\infty e^{-s\tau}\vartheta_1(x,\tau)d\tau \text{ and } \vartheta_2 = \int_0^\infty e^{-s\tau}\vartheta_2(x,\tau)d\tau$$

where $\bar{\vartheta}_1$ and $\bar{\vartheta}_2$ are the images of the functions (temperatures) $\vartheta_1(x, \tau)$ and $\vartheta_2(x, \tau)$ in the Laplace transforms and s is the parameter of the Laplace transform.

Then, the problem is written as

$$\frac{d^2\bar{\vartheta}_1}{dx^2} - q_1^2\bar{\vartheta}_1 = 0, -l < x < 0 \quad (15)$$