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Flow speeds and length scales in geodynamo models: The role of viscosity

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ABSTRACT

The geodynamo is the process by which turbulent flow of liquid metal within Earth's core generates our planet's magnetic field. Numerical simulations of the geodynamo are commonly used to elucidate the rich dynamics of this system. Since these simulations cannot attain dynamic similarity with the geodynamo, their results must be extrapolated across many orders of magnitude of unexplored parameter space. For this purpose, scaling analysis is essential. We investigate the scaling behavior of the typical length scales, ℓ, and speeds, U, of convection within a broad suite of geodynamo models. The model outputs are well fit by the scalings $\ell \propto E^{1/3}$ and $U \propto \mathcal{C}^{1/2} E^{1/3}$, which are derived from a balance between the influences of rotation, viscosity, and buoyancy (E is the Ekman number and $\mathcal{C}$ the convective power). Direct comparison with two previously proposed theories finds that the viscous scalings most favorably describe model data. The prominent role of viscosity suggested by these scaling laws may call into question the direct application of such simulations to the geodynamo, for which it is typically assumed that viscous effects are negligible.

1. Introduction

The Earth's magnetic field is generated by flowing liquid metal in its core. It is generally thought that this flow is driven by fluid buoyancy, as the core slowly cools and differentiates. The resulting convection generates electrical currents that maintain the geodynamo, and is subject to two significant forces: the Coriolis force, which stems from the Earth's daily rotation; and the Lorentz force, which accounts for the back-reaction of magnetic field on the flow from which it is generated. These key ingredients – field generation by rotationally constrained convection – are captured by self-consistent geodynamo models. Conditions in the core, however, are more extreme than any current simulation can possibly replicate (e.g., Wicht and Tilgner, 2010).

In particular, core fluid dynamics suffer from an extreme range of forces: Lorentz ~ Coriolis ≫ inertia ≫ viscosity. Estimates of the relative magnitudes of these forces can be quantified by non-dimensional parameters. The Rossby number characterizes the ratio between inertia and Coriolis forces in the core as $Ro \approx 10^{-6}$. The Ekman number quantifies the ratio between viscous and Coriolis forces in the core as $E \approx 10^{-15}$. Another extreme core parameter is the magnetic Prandtl number, which is the ratio between viscous and magnetic diffusion, $Pm \approx 10^{-6}$. These and other important dimensionless numbers are defined in Table 1. The smallness of these parameters leads to an extreme range of anticipated time and length scales important for core dynamics, which cannot be resolved in present-day simulations (Davies et al., 2011).

Since simulations are incapable of reaching the parameters necessary for true dynamic similarity with the geodynamo, we turn to scaling laws. Scaling laws depict the general behavior of one parameter with respect to others within a particular dynamical regime. There are two general purposes for developing and testing scaling laws. First, comparing theoretically founded scaling laws with observations in nature, experiments, or simulations tests our understanding of the basic physical processes responsible for producing these observations. Second, extrapolation of well-tested scaling laws, even without firm theoretical basis, permits predictions of phenomena we cannot directly observe.

Beginning with Glatzmaier and Roberts (1995), simulations of the geodynamo have proliferated such that there now exists an extensive population of results that permits the systematic scaling of their behavior (e.g., Christensen and Aubert, 2006; Olson and Christensen, 2006; Christensen, 2010; King et al., 2010; Jones, 2011). Here, we apply to such dynamo models a theoretical scaling law for average convective flow speeds. The scaling law, recently proposed for simpler, non-magnetic convection simulations (King et al., 2013), is based on a steady state balance between buoyant

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Table 1
Relevant dimensionless numbers, estimates of values in Earth's core, and the range of values reached in numerical simulations. Dimensional quantities are ν, viscosity; Ω, angular rotation rate; D, shell thickness; ρ'/ρ_0, fractional density anomaly; g, gravity; κ, thermal diffusivity; η, magnetic diffusivity; $\mathcal{P}$, convective power; A, outer boundary surface area; U, typical flow speed; B, magnetic field strength; μ_0, permittivity of free space. An estimate of Pr for the core depends on whether the buoyancy source considered is thermal ($Pr<1$) or compositional ($Pr>1$). An estimate of Ra in the core depends on the poorly constrained superadiabatic density contrast between inner core and mantle, and so a value is not specified.

Symbol	Name	Definition	Core	Simulations
E	Ekman	$\nu/\Omega D^2$	10^{-15}	$10^{-6} \le E \le 10^{-3}$
Ra	Rayleigh	$\rho' g D^3/\rho_0 \nu\kappa$		$3\times10^5 \le Ra \le 2.2\times10^9$
Pr	Prandtl	ν/κ		$0.1 \le Pr \le 30$
Pm	Magnetic Prandtl	ν/η	10^{-6}	$0.06 \le Pm \le 20$
$\mathcal{C}$	Convective power	$\mathcal{P}D^3/A\rho_0\nu^3$	10^{31}	$3\times10^4 \le \mathcal{C} \le 6\times10^{10}$
Re	Reynolds	UD/ν	10^9	$10 \le Re \le 2000$
Ro	Rossby	$U/2\Omega D$	10^{-6}	$10^{-4} \le Ro \le 0.8$
Λ	Elsasser	$B^2/\rho_0\mu_0\eta\Omega$	1	$0.03 \le \Lambda \le 300$

energy production and viscous dissipation. The important influence of rotation is revealed through its selection of the typical length scales of convective cells, which is a critical factor in setting the dissipation rate. Following King et al. (2013), we present hydrodynamic scaling laws for typical speeds and length scales of convection, U and ℓ, in Sections 3 and 4, and test these predictions against a suite of geodynamo simulations, which are introduced in Section 2. In Section 5, we compare these results against other proposed scaling laws. Finally, in Section 6, we discuss the implications of the preceding results.

2. Numerical dynamo models

The numerical dynamo model outputs analyzed here come from a suite of simulations carried out by Christensen using the MagIC numerical model (Christensen et al., 1999; Christensen and Aubert, 2006; King et al., 2010). In the model, the governing equations (momentum conservation equation, magnetic induction equation, heat advection–diffusion equation, and mass conservation) are evolved using the spectral transform method of Glatzmaier (1984) in a spherical shell with Earth-core-like geometry. Conditions enforced on inner and outer boundaries are constant temperature and zero flow. The outer boundary is electrically insulating, and, for almost all of the models used here, the inner core is also taken to be an insulator. The data set used here is identical to the “dynamo factory models” of King et al. (2010), consisting of 159 individual models, many of which were also used for scaling analysis by Christensen and Aubert (2006), Olson and Christensen (2006), and Christensen (2010). The parameter ranges accessed by the dynamo suite are given in Table 1.

Typical flow speeds are calculated as

$$U=\overline{\langle \boldsymbol{u}^2\rangle^{1/2}}, \tag{1a}$$

where $\boldsymbol{u}$ is the fluid velocity, angled brackets represent averages over the entire spatial domain, and overlines represent averages in time. Characteristic length scales of flow are calculated as the mean scale for kinetic energy (Christensen and Aubert, 2006)

$$\ell=\overline{\frac{\pi\langle \boldsymbol{u}^2\rangle}{\sum l\langle \boldsymbol{u}_l^2\rangle}}D, \tag{1b}$$

where $\boldsymbol{u}_l$ is the velocity at harmonic degree l, and D is the shell thickness (see Table 1).

3. Flow speeds: the mean kinetic energy equation

The equation governing fluid momentum in a rotating reference frame for Boussinesq magnetoconvection is

$$\underbrace{\rho_0(\partial_t \boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u})}_{\text{inertia}}+\underbrace{2\rho_0\boldsymbol{\Omega}\times\boldsymbol{u}}_{\text{Coriolis}}=\underbrace{-\nabla P}_{\text{pressure}}+\underbrace{\rho'\boldsymbol{g}}_{\text{buoyancy}}+\underbrace{\boldsymbol{J}\times\boldsymbol{B}}_{\text{Lorentz}}+\underbrace{\rho_0\nu\nabla^2\boldsymbol{u}}_{\text{viscosity}}, \tag{2}$$

where ρ_0 is the mean fluid density, $\boldsymbol{\Omega}$ is the rotation vector, P is the pressure (which includes hydrostatic gravitation and the centrifugal force), ρ' is the density anomaly, $\boldsymbol{g}=g\hat{\boldsymbol{r}}$ is the gravitational acceleration, $\boldsymbol{J}$ is the electrical current density, $\boldsymbol{B}$ is the magnetic field, and ν is the viscous diffusivity. The labels under each term in (2) will be used to identify them below. The system is considered Boussinesq in that density, $\rho(\boldsymbol{r},t)=\rho_0+\rho'(\boldsymbol{r},t)$, is treated as variable only in its contribution to the buoyancy force.

A particularly simple but exact relation for Boussinesq convection is the mean kinetic energy equation. Produced by the scalar product of velocity $\boldsymbol{u}$ with the momentum equation (2), averaging in time and integrating over the fluid volume V, the mean kinetic energy equation is

$$\int_V\left(\overline{u_r\rho' g}+\frac{1}{\mu_0}\overline{\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}}-\rho_0\nu\overline{\boldsymbol{\omega}^2}\right)dV=0, \tag{3a}$$

where u_r is the radial component of velocity and $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ is the vorticity. This particular form of the mean kinetic energy equation assumes no-slip boundaries. According to (3a), kinetic energy is produced by buoyant power

$$\mathcal{P}=\int_V\overline{u_r\rho' g}\,dV, \tag{3b}$$

by a generic source(s) of buoyancy, which is expended by the Lorentz work

$$Q_L=-\int_V\frac{1}{\mu_0}\overline{\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u}}\,dV, \tag{3c}$$

and viscous dissipation

$$\epsilon_\nu=\int_V\rho_0\nu\overline{\boldsymbol{\omega}^2}\,dV. \tag{3d}$$

Soderlund et al. (2012) argue that magnetic fields have a secondary influence on convective flow speeds in geodynamo models, citing a relatively weak Lorentz force in several simulations (which are similar to the simulations considered here). We follow from this suggestion and assume that Lorentz forces are unimportant for the typical magnitude of convective flow speeds in the models. The validity and geophysical relevance of this assumption are discussed in Section 6. Thus, in order to scale the flow speed, kinetic energy production is balanced with viscous dissipation

$$\mathcal{P}\sim\epsilon_\nu \tag{4}$$

Vorticity is taken to scale as U/ℓ, such that ϵ_ν can be scaled as $\rho_0\nu U^2V/\ell^2$. The viscous balance (4) then suggests flow speeds scale as

$$U\sim\left(\frac{\mathcal{P}d^2}{\nu\rho_0D^3}\right)^{1/2}. \tag{5}$$

In order to test this scaling against model outputs, we revert to dimensionless quantities for purely thermal convection. Total convective power $\mathcal{P}$ is related to the convective heat flow $\mathcal{P}_T$ by

$$\mathcal{P}_T=\rho_0c_PA\overline{u_rT'}\approx\frac{c_P}{\alpha gD}\mathcal{P}, \tag{6a}$$

where c_P is the specific heat, A is the surface area through which the heat power is fluxed, T' is the anomalous temperature, and α is the thermal expansivity. We make $\mathcal{P}$ non-dimensional by