Computers and Mathematics with Applications 70 (2015) 2401–2413

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A 66 line heat transfer finite element code to highlight the dual approach

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ARTICLE INFO

Article history:
Received 12 December 2014
Received in revised form 13 May 2015
Accepted 7 September 2015
Available online 16 October 2015

Keywords:
Finite element
Duality
Heat conduction
Free code
Convergence
Matlab code

ABSTRACT

The paper presents a compact Matlab implementation of a finite element code performing dual analysis of heat conduction problems. After a short presentation of the concerned variational principles, the conductivity matrices of the dual models are developed explicitly. In the next step, a Matlab procedure is written. It shows in 66 statements how it is possible to perform the dual analysis using the unique formalism based on a global conductivity matrix. This fully general method allows using the same finite element code to run the dual analyses. After showing the convergence properties, a second application is developed in order to show how easily the code can be modified. The last part of the paper, addressed to skilled readers, provides useful comments on the heat problem formulation. The Matlab procedure is proposed as a free code for educational purposes and can be extracted directly from the document using the standard copy and paste function.

1. Introduction

The dual approach to the finite element method consists in discretizing two conjugated fields of the concerned physical problem in order to obtain approximations that enclose the unknown exact solution. Although rather ancient, this approach has often been misunderstood or restricted to a small community of specialists. The main objective of this publication is to make the concept clear to most users of the finite element method (see Table 1 for notations).

Dual approach is also motivated by the characteristics of the problems to be addressed. Because it is appropriate to use radiosity methods [1] in the radiative heat transfer [2], when the heat conduction in solids is analyzed, it is practical to use a method of resolution based on the discretization of the same faces as in the radiosity methods.

Therefore, in the calculations of thermal conduction, while the most used methods are up to now the nodal ones [3], for the sake of integration with thermomechanical calculations, we prefer the finite element methods [4]. In this perspective, the thermal conduction elements that are best suited to the radiosity calculations are those where we are discretizing not the temperature field, which is a nodal variable, but the heat flux, which is a facet variable. This is the main reason for our choice of the dual approach in the domain of thermal conduction.

To solve the thermal problem, it is sufficient to discretize two of the following four fields: temperature, temperature gradient, stream function and heat flow. By minimizing the energy of the system with respect to the parameters of the chosen approximation, we obtain as integrals the equations that must meet the conjugate field. In this way, when discretizing the temperature field, the energy minimization provides a weak form (integral) of the equilibrium equations of the heat flow.

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http://dx.doi.org/10.1016/j.camwa.2015.09.007

Table 1
Taxonomy and dimensions.

Variable	Symbol	Units
Conductivity coefficient and tensor	k & k_{ij}	$\mathrm{W\,K^{-1}\,m^{-1}}$
Resistivity coefficient	r	$\mathrm{W^{-1}\,K\,m}$
State variable temperature	T	K
Temperatures array	t	K
Temperature gradient	g_i	$\mathrm{K\,m^{-1}}$
Temperature gradient array	G	$\mathrm{K\,m^{-1}}$
Thermal loads array	f	W
Heat flux and heat flux array	$q_i,\ p$	$\mathrm{W\,m^{-2}}$
Stream function	ψ	W
Connection matrix	B	$\mathrm{m^2}$
Conductivity matrix	C	$\mathrm{W\,K^{-1}}$
Core resistivity matrix	R	$\mathrm{W^{-1}\,K\,m^4}$
Dissipation function	I, J, F, P, Q	W K
Geometric variable: thickness	e	m
Geometric variable: length	a, b, c	m
Geometric variable: area	$\partial_i D$	$\mathrm{m^2}$
Geometric variable: volume	D	$\mathrm{m^3}$

The main advantage of the dual analysis is to perform with the same finite element mesh two analyses that enclose the exact solution [5]. After defining one of the two models, we can automatically build the second model by simply "translating" the data of the first one.

The objective of this report is to present the dual analysis in a simple and didactic way so as to make it accessible to a student who will be able to program it and test all its features. To meet this objective, we propose a short program written in Matlab® formalism. This approach was followed by O. Sigmund [6] to highlight the properties and the simplicity of the topology optimization method in order to make it accessible to most students.

Using a formulation in which the temperature and the flow are discretized, we develop two types of finite elements in the same formalism based on conductivity. So, it is possible to obtain the dual solutions, the first one with a strong knowledge of temperature and a weak one of the flow, the second one with a weak knowledge of the temperature and a strong one of the flow. This procedure requires only 66 lines of Matlab code.

The proposed formulation can be used to emphasize the properties of the dual analysis for academic problems, but it can also be improved in order to tackle more complex forms with use of unstructured meshes. It is also possible to improve the performance of the solver by taking into account the symmetry and the sparse character of the conductivity matrix.

The treatment of the convection boundary conditions does not involve particular problem. As well as the calculations in three dimensions, the consideration of the time aspect is fundamental in the simulation of thermal behavior in most applications. However, for the sake of conciseness and because their inclusion is not essential for the understanding of the process, we have chosen not to address these three issues, which are widely accessible in the literature [7].

2. Two fundamental functionals

2.1. Temperature

The following development is based on a publication dealing with the thermal application of the dual analysis [8]. The stationary (or permanent) heat conduction is one of the many field problems that can accept a variational formulation. We first introduce the basic formulation in terms of the temperature field alone. The finite element discretization uses parametric approximations of the temperature that ensure its continuity across the element interfaces.

In a simply connected domain and in the simplest formulation, in which we do not take into account the convection boundary conditions or the internal heat sources, the method consists of minimizing the temperature functional (in the next formula, the Einstein summation convention is applied: repeated indices mean summation). The following variables are introduced: the symmetric tensor of thermal conductivity k_{ij}, an imposed surface heat flux source $\bar{q}$ and an outside imposed temperature $\bar{T}_e$.

$$I(T) = \int_D \frac{1}{2} k_{ij} \partial_i T \, \partial_j T \, dD + \int_{S_2} \bar{q} T \, dS, \quad \text{extremum}. \tag{1}$$

The boundary S of the domain involves two parts, each one being possibly disjoint.

$$S = S_1 + S_2. \tag{2}$$

Two usual boundary conditions are related to these zones (the bars indicate that the quantity is imposed; n_i is the normal to the boundary surface S):

$$\begin{aligned} &T = \overline{T}_e \quad \text{on } S_1 \\ &n_i k_{ij} \partial_j T + \bar{q} = 0 \quad \text{on } S_2. \end{aligned} \tag{3}$$