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# Analysis and computation of the elastic wave equation with random coefficients

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ABSTRACT

We consider the stochastic initial–boundary value problem for the elastic wave equation with random coefficients and deterministic data. We propose a stochastic collocation method for computing statistical moments of the solution or statistics of some given quantities of interest. We study the convergence rate of the error in the stochastic collocation method. In particular, we show that, the rate of convergence depends on the regularity of the solution or the quantity of interest in the stochastic space, which is in turn related to the regularity of the deterministic data in the physical space and the type of the quantity of interest. We demonstrate that a fast rate of convergence is possible in two cases: for the elastic wave solutions with high regular data; and for some high regular quantities of interest even in the presence of low regular data. We perform numerical examples, including a simplified earthquake, which confirm the analysis and show that the collocation method is a valid alternative to the more traditional Monte Carlo sampling method for approximating quantities with high stochastic regularity.

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## 1. Introduction

The elastic wave equation describes phenomena such as seismic waves in the earth and ultrasound waves in elastic materials. It is a system of linear second order hyperbolic partial differential equations (PDEs) in a two or three dimensional physical space and has a more complex form than the standard acoustic wave equation, as it accounts for both longitudinal and transverse motions. There can also be surface waves traveling along a free surface, as well as waves that travel along internal material discontinuities.

It is often desirable to include uncertainty in the PDE models and quantify its effects on the predicted solution or other quantities of physical interest. The uncertainty may be either due to the lack of knowledge (systematic uncertainty), or due to inherent variations of the physical system (statistical uncertainty). In earthquake modeling, for instance, seismic waves propagate in a geological region where, due to soil spatial variability and the uncertainty of measured soil parameters, both kinds of uncertainties are present.

Probability theory provides an effective tool to describe and propagate uncertainty. It parametrizes the uncertain input data either in terms of a finite number of random variables or more generally by random fields. Several techniques are available for solving PDEs in probabilistic setting. The most frequently used technique is the Monte Carlo sampling which

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features a very slow convergence rate, see e.g. [1]. The slow convergence of Monte Carlo can be improved by quasi Monte Carlo (see, e.g., [2]) and multi-level Monte Carlo (see, e.g., [3]) techniques. Other recent approaches, which in certain situations feature a much faster convergence rate, include Stochastic Galerkin [4–8] and Stochastic Collocation [9–12]. Such methods are based on global polynomials and exploit the possible regularity that the solution might have with respect to the input parameters to yield a very fast convergence.

For stochastic elliptic and parabolic problems, under particular assumptions, the solution is analytic with respect to the input random parameters [9,13,14]. Consequently, Stochastic Galerkin and Stochastic Collocation methods can be successfully applied to such problems due to the fast decay of the error as a result of the high stochastic regularity. For stochastic hyperbolic problems, the regularity analysis is more involved. For the one-dimensional scalar advection equation with a time- and space-independent random wave speed, it is shown that the solution possess high regularity provided the data live in suitable spaces [15–17]. The main difficulty, however, arises when the coefficients vary in space or time. Recently, in [18], we have studied the second order acoustic wave equation with discontinuous random wave speeds. We have shown that unlike in elliptic and parabolic problems, the solution to hyperbolic problems is not in general analytic with respect to the random variables. Therefore, the rate of convergence may only be algebraic. However, a fast rate of convergence is still possible for some quantities of interest and for the wave solution with particular types of data. For the more difficult case of stochastic nonlinear conservation laws, where the corresponding regularity theory is lacking, we refer to the computational studies in [19–23].

In this work, we consider the elastic wave equation in a random heterogeneous medium with time-independent and smooth material properties, augmented with deterministic initial data and source terms and subject to homogeneous Neumann or Dirichlet boundary conditions. Since the analysis of hyperbolic problems with non-homogeneous boundary conditions and general boundary data is very complicated and not well developed (see [24,25] for the analysis for particular types of boundary conditions), we assume that the initial data and external forcing are compactly supported inside the domain and that the wave solution does not reach the boundary. In the numerical tests however, we employ absorbing boundary conditions and study the general case where the wave solutions can reach the boundary. Moreover, we are mainly interested in low-to-moderate frequency seismic waves propagating in slowly varying underlying media. Therefore we further assume that the wave length is not very small compared to the overall size of the domain and is comparable to the scale of the variations in the medium (see [26], where we consider multiscale elastic problems in highly varying media). We study the well-posedness and stochastic regularity of the problem by employing the energy method, which is based on the weak formulation of the problem and integration by parts. The main results of this paper, presented in Theorems 3, 4, 5, 6, is that the regularity of the solution or the quantity of interest in the stochastic space is closely related to the regularity of the deterministic data in the physical space and the type of the quantity of interest. We demonstrate that high stochastic regularity is possible in two cases: for the elastic wave solutions with high regular data; and for some high regular physical quantities of interest even in the presence of low regular data. For such problems, a fast spectral convergence is therefore possible when a stochastic collocation method is employed. We note that the stochastic collocation technique presented here is not new and has been widely used for the uncertainty propagation of many PDE models, see e.g. [9–12]. Here, we have adapted and employed the method to compute the statistics of the solution and QoIs and verify the regularity results obtained in the paper. The main contribution of this paper on the computational side is the inclusion of the filtering technique in the stochastic collocation algorithm. By introducing a low-pass filter, we add extra stochastic regularity, which in turn reduces the stochastic collocation error, as shown in this paper. We note that this will also introduce an extra filtering error. However, the filtering error can be suppressed by the collocation error if the filter parameters are properly chosen. A rigorous error analysis for the filtered quantities and the optimal choice of filter parameters will be studied in future works. We refer to [26] which – motivated by the present paper – studies the filtering technique in more details and presents some preliminary results.

This paper contributes to both *analysis* and *computation* of wave propagation problems subject to uncertainty. The main contributions include: (1) Extension of our work [18] to hyperbolic systems with vector-valued solutions; (2) Rigorous well-posedness and stochastic regularity analysis for the wave solution and mollified and filtered QoIs under general assumptions on the data, which is not discussed in other works; and (3) Application of the stochastic collocation method together with filtering techniques to seismic wave problems.

The outline of the paper is as follows. In Section 2 we formulate the mathematical problem and establish the main assumptions. The well-posedness of the problem is studied in Section 3. In Section 4, we provide regularity results on the solution and some physical quantities of interest. The collocation method for solving the underlying stochastic PDE and the related error convergence results are addressed in Section 5. In Section 6 we perform some numerical examples. Finally, we present our conclusions in Section 7.

## 2. Problem statement

Let $D$ be an open bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\partial D$, and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here, $\Omega$ is the set of outcomes, $\mathcal{F} \subset 2^{\Omega}$ is the $\sigma$-algebra of events and $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the stochastic initial–boundary value problem (IBVP): find a random vector-valued function $\mathbf{u} : [0, T] \times \bar{D} \times \Omega \to \mathbb{R}^d$, such that $P$-almost everywhere in $\Omega$, i.e. almost surely (a.s), the following holds:

$$\nu(\mathbf{x}, \omega)\, \mathbf{u}_{tt}(t, \mathbf{x}, \omega) - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}(t, \mathbf{x}, \omega)) = \mathbf{f}(t, \mathbf{x}) \quad \text{in } [0, T] \times D \times \Omega, \tag{1a}$$