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A two-grid mixed finite element method for a nonlinear fourth-order reaction–diffusion problem with time-fractional derivative☆

Yang Liu[a], Yanwei Du[a], Hong Li[a], Jichun Li[b,*], Siriguleng He[a]

[a] School of Mathematical Sciences, Inner Mongolia University, Hohhot, 010021, China
[b] Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV 89154-4020, USA

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ABSTRACT

In this article, we develop a two-grid algorithm based on the mixed finite element (MFE) method for a nonlinear fourth-order reaction–diffusion equation with the time-fractional derivative of Caputo-type. We formulate the problem as a nonlinear fully discrete MFE system, where the time integer and fractional derivatives are approximated by finite difference methods and the spatial derivatives are approximated by the MFE method. To solve the nonlinear MFE system more efficiently, we propose a two-grid algorithm, which is composed of two steps: we first solve a nonlinear MFE system on a coarse grid by nonlinear iterations, then solve the linearized MFE system on the fine grid by Newton iteration. Numerical stability and optimal error estimate $O(k_{\Delta}^{2-\alpha}+h^{r+1}+H^{2r+2})$ in L^2-norm are proved for our two-grid scheme, where k_{Δ}, h and H are the time step size, coarse grid mesh size, and fine grid mesh size, respectively. We implement the two-grid algorithm, and present the numerical results justifying our theoretical error estimate. The numerical tests also show that the two-grid method is much more efficient than solving the nonlinear MFE system directly.

1. Introduction

In this article, we consider solving a nonlinear fourth-order reaction–diffusion problem with time-fractional derivative

$$\frac{\partial u}{\partial t}-\frac{\partial^{\alpha}\Delta u}{\partial t^{\alpha}}-\Delta u+\Delta^2 u+f(u)=g(\mathbf{x},t),\quad (\mathbf{x},t)\in\Omega\times J, \tag{1.1}$$

which satisfies the boundary condition

$$u(\mathbf{x},t)=\Delta u(\mathbf{x},t)=0,\quad (\mathbf{x},t)\in\partial\Omega\times\bar{J}, \tag{1.2}$$

and initial condition

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{1.3}$$

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* Corresponding author.
E-mail address: jichun.li@unlv.edu (J. Li).

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where Ω is a bounded Lipschitz polyhedron of R^d ($d = 2$, or 3) with boundary $\partial\Omega$, and $J = (0, T]$ is the time interval. The source term $g(\mathbf{x}, t)$ and the initial function $u_0(\mathbf{x})$ are some given functions, the nonlinear reaction term is $f(u) = u^3 - u$, and $\frac{\partial^\alpha p(\mathbf{x},t)}{\partial t^\alpha}$ with $\alpha \in (0, 1)$ is defined by the following Caputo fractional derivative

$$\frac{\partial^\alpha p(\mathbf{x}, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial p(\mathbf{x}, \tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}. \tag{1.4}$$

The fourth-order reaction–diffusion problems appear in many scientific fields, e.g., traveling waves of reaction–diffusion systems, propagation of domain walls in liquid crystals and pattern formation of bistable systems. Many numerical methods have been developed for solving the time-dependent fourth-order reaction–diffusion equations, cf. [1,2] and references cited therein. In [1], a mixed finite element (MFE) method based on the backward Euler method is studied and a priori error analysis is given. In [2], the optimal error bounds for a MFE method based on a special interpolation operator have been obtained for a fourth-order reaction–diffusion problem and a fourth-order elliptic equation. Based on the standard fourth-order reaction–diffusion problem [2,1,3], the nonlinear time-fractional fourth-order reaction–diffusion system (1.1) with α-order is proposed. It is easy to see that when α goes to 0, Eq. (1.1) is reduced to a standard fourth-order reaction–diffusion equation. On the other hand, when α is close to 1, the problem (1.1) can be regarded as a fourth-order reaction–diffusion system with viscous term.

In recent years, developing various numerical algorithms for solving fractional partial differential equations (FPDEs) has received much attention. The popular numerical methods for FPDEs include finite element methods [4–15], finite volume/element methods [16–18], finite difference methods [19–23,16,24–33], discontinuous Galerkin methods [34–37], and spectral methods [38–42], etc. To our best knowledge, there exist only a few papers on numerical methods for the fourth-order FPDEs. In [36], a one-dimensional time-fractional fourth-order problem was solved by a local discontinuous Galerkin method. Recently, a linear time fractional fourth-order diffusion equation solved by the finite element method was investigated in [13]. Here, we develop and analyze a two-grid algorithm for solving the nonlinear time-fractional reaction–diffusion system (1.1).

The two-grid method was proposed by Xu [43,44]. In [43], a two-grid discretization technique based on finite element method was proposed for a semilinear elliptic boundary value problem. In [44], some finite element discretization techniques based on two (or more) subspaces for nonlinear elliptic partial differential equations were studied. The initial successful application of the two-grid technique for nonlinear elliptic problems inspired many further studies and applications of the two-grid methods. Dawson, Wheeler and Woodward [45] discussed a two-grid method based on MFE system for nonlinear parabolic equations. Xu and Zhou [46] solved the eigenvalue problems by using a two-grid discretization scheme. Marion and Xu [47] studied a two-grid method for solving semilinear parabolic equations. Chien and Jeng [48] applied a two-grid scheme for semilinear elliptic eigenvalue problems. Chen et al. [49], Chen et al. [50], Wu and Allen [51], and Liu et al. [52] gave some different analysis on two-grid algorithm using expanded MFE methods for some nonlinear partial differential equations. Chen and Chen [53] discussed a two-grid method for nonlinear reaction–diffusion equations by using MFE methods. Chen and Liu [54] presented a finite volume element method based on a two-grid algorithm for solving the second-order nonlinear hyperbolic equations. Bajpai and Nataraj [55] developed a two-grid finite element scheme combined with Crank–Nicolson scheme for the equations of motion arising in the Kelvin–Voigt model. In all, we are unaware of any work using the two-grid method for (1.1). The novelty and major achievement of this paper is that we successfully extend the two-grid method to solve the complicated two-dimensional time-fractional fourth-order problem by the MFE method. Optimal convergence rate in both time and space is proved.

In this article, we propose a two-grid algorithm for solving the nonlinear time-fractional fourth-order reaction–diffusion system (1.1) by using MFE method. We formulate a nonlinear fully discrete MFE system, where the integer derivative $\frac{\partial u}{\partial t}$ is approximated by a second order backward difference method, while the time-fractional derivative is approximated by a difference scheme with the convergence order $k_\Delta^{2-\alpha}$, and the spatial discretization is handled by using MFE method. The nonlinear coupled system is solved by using two-grid method. We first solve a nonlinear MFE system on a coarse grid, then solve a linearized MFE system on a fine grid. Numerical stability and error estimate are proved on both the coarse grid and fine grid. More specifically, we prove the optimal space–time convergence $O(k_\Delta^{2-\alpha} + h^{r+1} + H^{2r+2})$ in the L^2-norm. Finally, numerical results are presented to support our theoretical analysis.

The rest of the paper is organized as follows. In Section 2, we present a two-grid algorithm combined with the MFE scheme in the spatial direction and an implicit two step backward difference scheme in the temporal direction. In Section 3, we carry out the stability analysis for both the coarse grid and fine grid. In Section 4, we prove the optimal a priori error estimate for both the coarse grid and fine grid. In Section 5, we present the numerical results obtained by both the two-grid algorithm and the MFE method. In Section 6, we conclude the paper with some remarks.

Throughout this article, we denote $C > 0$ as a generic constant, which is independent of the fine grid mesh size h, coarse grid mesh size H, and the time step size k_Δ. Furthermore, we define the inner product in $(L^2(\Omega))^d$ by $(\cdot, \cdot)$ equipped with norm $\|\cdot\|$, and the standard Sobolev space $H^k(\Omega)$ (or $H_0^k(\Omega)$) equipped with norm $\|\cdot\|_k$.

2. The two-grid algorithm based on MFE method

To approximate both the integer and fractional derivatives in the temporal direction, we assume a uniform partition $0 = t_0 < t_1 < t_2 < \cdots < t_{N_{k_\Delta}} = T$ of the time interval $[0, T]$, where $t_n = nk_\Delta$, $n = 0, 1, \ldots, N_{k_\Delta}$. The time step size