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An efficient implicit spectral element method for time-dependent nonlinear diffusion equations by evaluating integrals at one quadrature point

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ABSTRACT

We present an implicit spectral element method to approximate the solution of time-dependent nonlinear diffusion equations in complex geometries. We propose a nodal expansion to approximate derivatives of an unknown function, so integrals involving nonlinearities are evaluated at one quadrature point and discrete form of nonlinear diffusion equations is obtained without forming local and global stiffness matrices. Using the method-of-lines (MoL), the nonlinear partial differential equation is reduced to a nonlinear system of ordinary differential equations such that the MoL formulation is derived in terms of the differentiation matrix instead of the stiffness matrix. Since the arising system is stiff, an implicit method is employed to solve it. Also, with a simple algorithm we obtain a closed analytical form for the Jacobian matrix of the system of nonlinear algebraic equations. In addition, the differentiation matrices for triangular elements and deformed quadrilateral and hexahedral elements are obtained. For deformed elements an isoparametric mapping is used where the expansion coefficients are obtained with the aid of the Gordon–Hall mapping. Several benchmarks, including the scalar and system of these equations, are carried out. As physical applications, the cell motion model over long time, the tumour angiogenesis model on a mixed quadrangle–triangle mesh, and two and three-dimensional *p*-Laplace equations for various choices of the parameter *p* are considered.

1. Introduction and motivation

In this paper, we consider the time-dependent nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (F(u, \nabla u)\nabla u) + f, \tag{1.1}$$

subject to appropriate boundary and initial conditions. Many applications of these equations exist e.g. image processing [1,2], biofilm model [3–5], tumour cell invasion [6,7], epidemic model [8], wound-healing angiogenesis [9], bacterial pattern formation [10], model of cell motility [11], and chemotaxis [12,13].

Finding analytical solutions of nonlinear PDEs is difficult or impossible. Alternatively, some semi-analytical methods [14] and numerical methods are available [15]. For some special cases of time-dependent nonlinear diffusion equations, an

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analytical solution exists [16–18]. But generally, we have to solve these equations numerically. Several numerical methods such as the finite difference [7,10,3,5,19], pseudospectral [12], finite volume [8,20,21], mixed finite element [22], dual reciprocity boundary element [23], and marker [24] have been proposed to approximate the solution of (1.1).

Also, we consider the nonlinear advection–diffusion–reaction PDE coupled with two ordinary differential equations (ODEs):

$$\frac{\partial u}{\partial t} = \nabla \cdot [\alpha \nabla u - u\,(F(w)\nabla w + \beta \nabla v)] + S(u), \tag{1.2}$$

$$\frac{\partial w}{\partial t} = G(u, w), \tag{1.3}$$

$$\frac{\partial v}{\partial t} = H(u, v), \tag{1.4}$$

where the diffusion coefficient α is small and β is a constant. In Eq. (1.2) the term $\nabla \cdot [-uF(w)\nabla w]$ behaves as a nonlinear diffusion term. Previously, the system (1.2)–(1.4) was solved by an explicit subparametric spectral element method-of-lines (MoL) [25] and adaptive finite element methodology [26]. The system (1.2)–(1.4) arises from the tumour angiogenesis model.

In this paper, we present an implicit spectral element approximation using the MoL to solve the time-dependent nonlinear diffusion equation (1.1) and system (1.2)–(1.4). The MoL is applied in order to use available methods for solving stiff initial value problems (IVPs).

The spectral element method is based on a weak form of a PDE. As a result, to solve the nonlinear diffusion equation only the continuity of the approximation at the interface nodes must be satisfied. The spectral element method was first proposed by Patera [27] to find the numerical solution of the incompressible Navier–Stokes equations. This method has been successfully used to solve PDEs in many areas of science and technology e.g., the Black–Scholes equation [28], Klein–Gordon equation [29], elastic wave [30], acoustic wave [31], seismic wave [32], moving-boundary problems [33], Maxwell equations [34], shallow water equations [35], Helmholtz equation [36], the P_N neutron transport equations [37], vector radiative transfer equation [38], predator–prey system modelling interacting populations [39], the Pennes bioheat transfer equation [40] and second-order nonlinear partial differential equations [41]. The spectral element method can also be implemented in terms of modern parallel architectures [42]. Extensive analysis of this method is investigated by Canuto et al. [43].

In brief, the spectral element method has three significant properties:

1. Feasibility of high order approximations,
2. Flexibility for complex geometries,
3. Cardinality property of Lagrangian basis functions.

Due to two first properties, the spectral element method has a capability of simulating complicated real-world problems. Third property helps us to implement the method easily and reduces computational cost significantly. The cardinality property of Lagrangian basis functions leads to some advantages to be discussed in detail below. First by having the cardinality property, the continuity of the approximation at the interface nodes is automatically satisfied [27,40,44]. In fact, the spectral element approximation is naturally continuous. As the spectral element approximation is composed of piecewise polynomials, so the continuity of this approximation yields the square integrability of its first derivative [45]. Second, the mass matrix is diagonal if the same set of points is used to construct the Lagrange polynomials and approximate the integrals [46]. The diagonal entries are determined by the values of the Jacobian of mapping and quadrature weights. Also due to the cardinality property, using tensor-product basis functions for deformed elements leads to matrix-free evaluations such that storage and work estimates for discretizations in R^d containing E deformed elements of order P take $O(EP^d)$ and $O(EP^{d+1})$ operations, respectively, [42]. Fourth, Dirichlet boundary conditions are easily imposed. The next advantage of the cardinality property appears to efficiently solve nonlinear reaction PDEs. In fact by using this property, the numerical evaluation of integrals involving nonlinear reaction terms is performed with $O(1)$ operation. Moreover, the discrete form of nonlinear reaction terms is easily obtained like the pseudospectral method based on the strong form of the PDE and the number of non-zero entries of the Jacobian matrix of the system of nonlinear algebraic equations is reduced [39].

In the spectral element method, it is usual to use a nodal expansion to approximate the unknown function and then take derivative from this expansion with respect to a space variable to find an expansion for the spatial derivative of the unknown function. However, the arising expansion for the spatial derivative is no longer a nodal one. To exploit the cardinality property of the Lagrangian basis functions for solving the nonlinear diffusion equation (1.1) or the system (1.2)–(1.4) we propose to use a nodal expansion for the spatial derivative. This trick results in a significant reduction of computational cost and also makes the spectral element method easy to implement.

To approximate the solution of the nonlinear diffusion equation (1.1) or the system (1.2)–(1.4) numerically, evaluating integrals may be complicated due to nonlinearities. Also, the stiffness and Jacobian matrices change in time.

Our novelties in this paper are:

1. Evaluating integrals involving nonlinearities at one quadrature point. Thus, cost of the numerical integration is reduced from $O(\mathcal{N})$ operations to $O(1)$ where $\mathcal{N}$ is the number of quadrature points.
2. Deriving discrete form of nonlinear diffusion terms without forming local and global stiffness matrices.