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Symmetric finite volume element approximations of second order linear hyperbolic integro-differential equations

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ABSTRACT

In this paper, based on barycenter dual meshes, we develop one semi-discrete and two full discrete symmetric finite volume element schemes for second order linear hyperbolic integro-differential equations. The optimal order error estimates in L^2 and H^1-norms are derived for the semi-discrete scheme. Numerical experiments confirm the performance of the symmetric schemes, and further show that the L^2-norm convergence rate of the full discrete backward Euler and Crank–Nicolson schemes to be $O(h^2 + \tau)$ and $O(h^2 + \tau^2)$, respectively.

1. Introduction

In this paper, we consider symmetric finite volume element (FVE) method for the following class of second order linear hyperbolic integro-differential initial boundary value problem: Find $u = u(\mathbf{x}, t)$ such that

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mathcal{A}\nabla u) - \displaystyle\int_0^t \nabla \cdot (\mathcal{B}\nabla u(s))ds = f(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded polygonal domain with boundary $\partial\Omega$, $\mathbf{x} = (x, y)$, $\mathcal{A} = \{a_{i,j}(\mathbf{x}, t)\}$ and $\mathcal{B} = \{b_{i,j}(\mathbf{x}, t, s)\}$ are 2×2 given symmetric and positive definite real-value matrices with smooth coefficients, and $f(\mathbf{x}, t) : \Omega \times [0, T] \longrightarrow R$. The initial functions u_0 and u_1 are assumed to be smooth enough so that the problem (1.1) has a unique solution in the Sobolev spaces. The problem of (1.1) is very important and it often arises from the modeling reactive flows or material with memory effect [1].

Finite volume element methods, also called generalized methods [2], covolume methods [3] or finite volume methods [4–7], have been widely used in many sciences and engineering fields, e.g., computational fluid mechanics, heat and mass transfer and petroleum reservoir simulations. The FVE methods are usually easy to be implemented and can offer the flexibility to handle complicated geometries and boundary conditions. More important, numerical solutions obtained by

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the FVE methods can preserve certain conservation property, which is not preserved for the finite element methods. For a recent survey on FVE methods, we refer the readers to [8].

Recently, FVE methods for parabolic integro-differential problems were widely considered by many researchers in [9–12]. In [10,11], the authors obtained L^2 and H^1-norms error estimates in the framework of the standard Petrov–Galerkin formulation, but the L^2-error estimate's regularity requirement seems to be too high compared with the finite element methods. Subsequently, Sinha et al. [12] improved the estimates for a new variant of the Ritz–Volterra projection, and then obtained the optimal-order L^2-error estimate for both smooth and nonsmooth initial data. However, FVE methods for hyperbolic integro-differential equations have not been much studied in the literature. The relevant work was found in [13] where Karaa et al. constructed both semidiscrete and completely discrete FVE schemes, then the optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms were obtained by the minimal regularity assumption on the initial data. In addition, the optimal error estimate was presented for an explicit completely discrete scheme. We refer the readers to [14–26] for the recently developments of hyperbolic and hyperbolic integro-differential problems by FVE methods.

As we have known, the property of symmetry represents the fundamental physical principle of reciprocity in many cases. Unfortunately, the standard FVE schemes usually generate a linear systems with asymmetric matrix [27,3]. This property of asymmetric will lead to the fact that many efficient methods for solving large linear algebraic equations, e.g., the conjugate gradient method, cannot be employed. In recent years, the authors in [4–7] have constructed and analyzed one class of symmetric finite volume schemes for the self-adjoint elliptic, parabolic and eigenvalue problems based on the triangulation and barycenter dual mesh. At the same time, based on circumcenter dual mesh, the authors in [28,29,9] have employed symmetric modified FVE schemes for the self-adjoint elliptic, parabolic, nonlinear parabolic and parabolic integro-differential problems, however there were lack of some efficient experiments. Based on quadrilateral grids, a symmetric FVE scheme was established for a class of elliptic problems by Shu et al. [30]. Further, for the nonlinear convection diffusion problems, the authors in [31] constructed a symmetric characteristic FVE scheme by biquadratic elements and multistep methods.

In this paper, we apply the same techniques in [4–7] to establish one semi-discrete and two full discrete symmetric FVE schemes for the hyperbolic integro-differential problem (1.1) based on barycenter dual partition. Then, the optimal order error estimates in L^2 and H^1-norms are derived for the semi-discrete schemes by the Ritz–Volterra projection in the context of the symmetric FVE method. Numerical experiments are presented to demonstrate the effectiveness of two full discrete symmetric schemes and further show that the L^2-norm convergence rate of the backward Euler and Crank–Nicolson schemes to be $O(h^2+\tau)$ and $O(h^2+\tau^2)$, respectively.

This paper is organized as follows: In Section 2, we introduce the necessary notations and preliminaries. Further, the semi-discrete symmetric FVE scheme is constructed and the error estimates in H^1 and L^2-norms are devoted in Section 3. In Section 4, two new full discrete backward Euler and Crank–Nicolson symmetric FVE schemes are established. In Section 5, a number of numerical experiments are reported to confirm the efficiency of the symmetric schemes and finally, we summarize our symmetric FVE schemes for the hyperbolic integro-differential equations in Section 6.

2. Notation and preliminaries

In this paper, we use the standard notation for the Sobolev space $W^{m,p}(\Omega)$ with $1 \le p \le \infty$ where the norm on $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{m,p,\Omega} = \|u\|_{m,p} = \begin{cases} \left(\sum_{\|\alpha\| \le m} \|D^\alpha u\|_{L^p}^p \right)^{1/p}, & \text{for } 1 \le p < \infty, \\ \max_{\|\alpha\| \le m} \|D^\alpha u\|_{L^\infty}, & \text{for } p = \infty, \end{cases}$$

and the semi-norm

$$|u|_{i,p,\Omega} = |u|_{i,p} = \left(\sum_{|\alpha|=i} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}, \quad \text{for } 1 \le p < \infty,$$

with the standard modification for $p = \infty$. In particular, when $p = 2$, the Sobolev space $W^{m,2}$ is written by H^m and its norm is denoted by $\|\cdot\|_m$.

Following [32], the Sobolev space $W^{m,p}(0, T; X)$ is defined as follows

$$W^{m,p}(0, T; X) := \{u : (0, T) \longrightarrow X \,|\, \|D_t^i u\|_X \in L^P(0, T), 0 \le i \le m\},$$

and its norm

$$\|u\|_{W^{m,p}(0,T;X)} = \|u\|_{W^{m,p}(X)} := \sum_{i=0}^{m} \left(\int_0^T \|D_t^i u\|_X^p dt \right)^{1/p},$$

with the standard modification for $p = \infty$, where $D_t = \partial/\partial t$, $D_t^i = \partial^i/\partial t^i$, $i = 2, \ldots, m$. For $m = 0$, the Sobolev space $W^{m,p}(0, T; X)$ is simplified by $L^p(X)$.