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Non-monotonic traveling wave and computational solutions for gas dynamics Euler equations with stiff relaxation source terms

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ABSTRACT

We study the existence of non-monotone traveling wave solutions and its properties for an isothermal Euler system with relaxation describing the perfect gas flow. In order to confront our results, we first apply a mollification approach as an effective regularization method for solving an ill-posed problem for an associated reduced system for the Euler model under consideration, which in turn is solved by using the method of characteristics. Next, we developed a cheap unsplitting finite volume scheme that reproduces the same traveling wave asymptotic structure as that of the Euler solutions of the continuous system at the discrete level. The method is conservative by construction and relatively easy to understand and implement. Although we do not have a mathematical proof that our designed scheme enjoys the asymptotic preserving and well-balanced properties, we were able to reproduce consistent solutions for the more general Euler equations with gravity and friction recently published in the specialized literature, which in turn are procedures based on a Godunov-type scheme and based on an asymptotic preserving scheme, yielding good verification and performance to our method.

1. Introduction

The system of Euler equations in gas fluid dynamics has been the focus of an intense discussion on many physical phenomena with many objectives in mind and ranging from mathematical theory to numerics with applications. We consider the Euler equations with gravity and a friction terms that describe an isentropic gas flow in a porous media model, see, e.g., [1–7]. Indeed, besides using the below full Euler system as motivation to design an unsplitting scheme for its approximation, we also focus on using a distinct isothermal-like Euler gas system linked to the friction relaxation term in order to study the existence of non-monotone traveling wave solutions and its properties for such stiff relaxation system; this will be depicted in details in Section 3.1 and continued in Section 5. Thus, the full governing Euler equations with gravity and friction is given by,

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ (\rho E+p)u\end{bmatrix}=\frac{1}{\epsilon}\begin{bmatrix}0\\ \rho(g-\alpha u)\\ \rho(gu-\alpha\varrho u^2)\end{bmatrix}, \tag{1}$$

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where ρ, u, p and E are, respectively, the density, the velocity, the pressure and the total energy and ϵ is the relaxation time scale. The two positive constants g and α denote the gravity number and the friction coefficient, respectively; the quantity ϱ is related to a compressible Euler flow [7,8]. Indeed, we will see that the friction coefficient α alone can be viewed as the relaxation time scale ϵ, associated to the underlying balance law system. The system (1) is closed by the classical definition of the total energy $E = e + u^2/2$, where e is the internal energy, and by a pressure law $p = p(\rho, e)$, which in turn satisfies classical gas dynamic assumptions [8].

For instance, we mention the recent works with contributions on the subject [1–8] in which is described the phenomenon of friction of gases, including an interesting connection with a parabolic Darcy-like equation in porous media transport phenomena [7]. In these works the friction term acts as a dynamic damper effect over the solution. If the friction term is sufficient large, the source term balance the flux term leading to a asymptotically stable equilibrium state at a reference numerical value [2,8]. In gas dynamics, in industrial problems, the dominant physical phenomenon is the pressure loss due to pipe wall friction effects [6]. Thus, this is also relevant in real world applications in engineering. For concreteness in [2], the authors developed a numerical approach for modeling the interaction of an isentropic gas dynamics in a porous medium to the Euler equations, describing specifically the numerical coupling of the dynamic transition between low and high friction regions. The numerical procedure is a finite volume scheme based on upwinding the source term at the interface of the rock. In [3,4] the authors considered a more general gas dynamics problem for Euler equations, taking into account gravity, friction and compressibility. They introduced a Godunov-type scheme based on a Riemann solver, which in turn exhibits the properties well-balanced and asymptotic preserving. Indeed, the numerical illustrations reported there show that scheme behave quite well on test problems. It has long been recognized that upwind differencing like that of Godunov-type schemes are able to resolve shocks sharply without spurious oscillations exhibiting good properties since Riemann solvers have exact solutions and the sequence of numerical flux calculations are well-controlled approximations. On the other hand, the main drawbacks of this procedure is just the calculation cost of such "exact" solutions as well as to the fact that its extension to multiple dimensions might be sensibly hard. For instance, in [1,3,4] a good knowledge of the associated solution of the Riemann problem is deeply required for implementation of the method. It is worth mentioning that the numerical method proposed here does not require the use of Riemann solvers and it seems to be able to reproduce the numerical results of test problems discussed in [2–5] without the need to be forced to analyze the equations in an intricate level of details, at least in a first attempt for its investigation. Our motivation to this is sutil but relevant, due to the connection between Euler equations [2–5] and the porous media equation [7], and is also related to previous works of the authors [9,10], where two distinct approaches were used to address issues related to the systems of equations modeling flow in porous media for which the theory is not well understood. Of course, all methods exhibit advantages and disadvantages since the issue of balance laws is definitely a very hard problem with a lack of general theory. Thus, we combine new successful ideas (to the best of our knowledge) from central differencing schemes [9] and from asymptotic techniques for balance laws with stiff sources [10] in order to design, based on this new approach, a cheap unsplitting finite volume scheme that reproduces the same traveling wave structure as that of the Euler solutions of the continuous system at the discrete level without any a priori knowledge of the inherent character of the stiff balance law under consideration. In spite of the subject of the latter works, we considered a modified Euler equation with relaxation source term describing a distinct phenomenon.

As a concrete example, we consider the Euler equations with a relaxation term driving the temperature towards a reference constant temperature value [11,8], which in turn is conducive to our analysis. This is precisely the isothermal flow equation for gas dynamics in a one-dimensional tube surrounded by a recipient at constant temperature. It is assumed that the heat flows in or out of the recipient instantaneously, so that a constant temperature is maintained; see [11,8] for more details. The relaxation phenomenon is associated with an energy balance to which the system reaches equilibrium.

We study the existence and asymptotic behaviors of the non-monotonic traveling wave solutions connecting two equilibrium states associated to the energy balance equation. We use the method of outer expansion analysis in the spirit of the Chapman–Enskog expansion for kinetic equations. Our asymptotic analysis on the isothermal Euler flow problem shows that stiff relaxation plays the role of self-sharpening effect in the nonlinear balance between source and flux with respect to the small relaxation time, provided that suitable scale conditions hold. We first apply a mollification approach as an effective regularization method for solving an ill-posed problem for an associated reduced system for the Euler model under consideration, which in turn is solved by using the method of characteristics. Such approximate solutions are used to prove the existence of traveling wave solutions, and investigate the asymptotic behavior of the traveling waves in relation to various interacting parameters of the system. Indeed, we obtain an asymptotic description of the solution for large t in relation to the interacting parameters, namely friction and diffusion, and show the existence of non-monotonic traveling wave solutions connecting the two equilibria with such asymptotic rates. In this work it is also shown by asymptotic analysis and numerical examples a new formal way to study the behavior for small relaxation parameter regimes linked to friction and induced-diffusion regimes. We study the long time behavior of the waves, where the lower first order terms decay scaled by t. We do not prove the convergence of the solution, however we believe that this approach gives very good approximations for the outer expansion, which in turn is illustrated by several numerical test problems. Indeed, we developed a cheap unsplitting finite volume scheme that reproduces the same traveling wave asymptotic structure as that of the analytic Euler solutions of the continuous system at the discrete level. The method is conservative by construction and relatively easy to understand and implement. The scheme is based on central differencing schemes in the Nessyahu–Tadmor and Lax–Friedrichs framework, see, e.g., [9,12,13]. Numerical experiments are also presented in order to illustrate the theoretical results. We point out that we do not have a mathematical proof that our designed scheme possess the properties