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Numerical error control for second-order explicit TVD scheme with limiters in advection simulation

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ABSTRACT

This paper analyzes the causes of the numerical errors in terms of numerical diffusion and compression arising from the use of explicit second-order total variation diminishing (TVD) schemes in one-dimensional advection simulation. It demonstrates that different TVD limiters may have very different performances in different advection simulations, because of the so-called numerical diffusion and compression. The accuracy of the computed results is found to depend on not only the limiter functions themselves but also the advection features such as the concentration distribution, advection velocity and time step, etc. According to such relations, the effective ranges of the MIN_MOD, Van Leer and SUPERBEE limiters are characterized by introducing a dimensionless parameter which reflects the key features of advections, aiming to provide an approach to select a proper TVD limiter in advection simulation.

1. Introduction

Advection is a common physical process for mass, momentum and energy transport, e.g. air mass movement in the atmosphere [1,2], and transport of pollutant and sediment in environmental flows [3–5]. In numerical simulations, the advection is hard to cope with due to its hyperbolic property [6,7], and may induce numerical errors such as numerical diffusion and oscillations, especially near the discontinuous parts of the solution [8–10]. Generally speaking, numerical diffusion is a result of low order schemes, while numerical oscillations are attributed to the higher order schemes as limiters [11]. To preserve the accuracy and monotonicity of a scheme, such numerical errors must be controlled. In recent decades, some prevalent high resolution numerical schemes are developed, e.g. the flux-corrected transport (FCT) [12], the essentially non-oscillatory (ENO) scheme and weighted ENO scheme (WENO) [13], and the second-order total variation diminishing (TVD) scheme [14,15] which is also termed as the monotone upstream scheme for conservation law (MUSCL) [16–19]. Among these, the second-order TVD schemes are the most efficient ones by using limiter functions and attract most attention [20–24].

The second-order TVD schemes preserve the accuracy and monotonicity through the employment of limiters. Different limiters may have different performance in a specific advection case. For instance, a TVD scheme with a SUPERBEE limiter is able to capture a shock more accurately than that with a MIN_MOD limiter [25]. It is clear that a second-order TVD scheme

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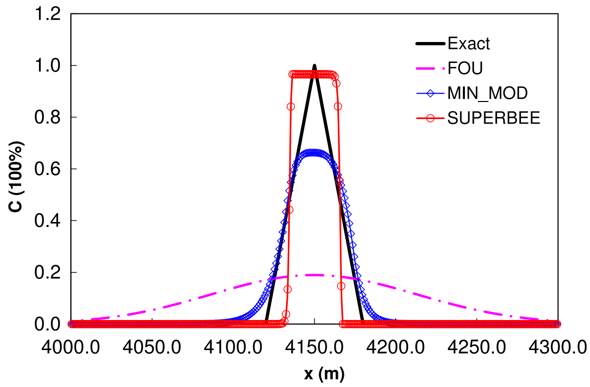

Fig. 1. Numerical errors caused by numerical diffusion and compression.

can suppress the numerical diffusion, however, numerical diffusion still exists in certain degree when using some limiters like the MIN_MOD one as illustrated in Fig. 1. Certain limiters may better control numerical diffusion for the same case but they may over suppress the numerical diffusion and thus give rise to numerical compression which may cause staircasing of oblique shocks [26]. Obviously, numerical compression is also numerical error leading to less accurate prediction, especially using explicit Euler methods to update the computation to a new time level (Fig. 1). Explicit Euler methods are broadly applied in solving the hyperbolic equations, for example in [27–29], because of its simplicity in implementation and efficiency in computation. In order to minimize the numerical distortion in terms of either numerical diffusion or numerical compression, an appropriate TVD limiter must be selected for a specific case. However, few research quantitatively analyzes such numerical distortions to provide a guideline to select a proper limiter function in advection simulation using second-order TVD schemes, although it is well known that the SUPERBEE and MIN_MOD limiters may be compressive and diffusive in certain cases, respectively.

This work investigates the relationships between the performances of the TVD limiters and the features of advections which include value's distribution, advection velocity, cell size and time step, etc. The conditions under which a limiter has the lowest numerical error for one-dimensional (1D) advection problems are analyzed according to the relationships. Then the effective ranges of three typical TVD limiters in 1D advection simulation are summarized. In this paper, the governing equations and the numerical scheme for advection simulation are briefly introduced in Section 2; numerical errors caused by the TVD scheme with different limiters are assessed in Section 3 and the applicable ranges for different TVD limiters are proposed here; the applicable ranges are verified by two relatively more complicated test cases in Section 4 and the conclusions are drawn in Section 5.

2. Governing equations and numerical schemes

As described in [8], the advection equation can be written as

$$\frac{\partial C}{\partial t} + \nabla \cdot (\mathbf{v}C) = 0, \tag{1}$$

where C denotes the variable being advected and $\mathbf{v}$ is the velocity vector. Using finite volume method (FVM), Eq. (1) is integrated over a control volume i with the volume of V as

$$\int_V \left(\frac{\partial C}{\partial t}\right) dV + \int_V \nabla \cdot (\mathbf{v}C) dV = 0. \tag{2}$$

The first term in Eq. (2) is treated using a finite difference method and the second term is rearranged through divergence theorem. The value of variable may be then updated to a new time level by an explicit Euler method as

$$C_i^{n+1} = C_i^n - \frac{\Delta t}{V} \sum_{j=1}^{k} (\mathbf{v}_j^n C_{ij}^n \cdot \mathbf{n}_j l_j), \tag{3}$$

in which n and k stand for the time step and the number of faces of the control volume, respectively. j represents the jth face of a cell, e.g. j is from 1 to 4 for a 2D rectangular cell. $\mathbf{n}_j$ is the outwards normal vector to the jth face. l_j represents the area in 3D or length in 2D of the jth face. The face variable C_{ij}^n is computed by a TVD scheme explicitly as

$$C_{ij}^n = C_{i-1}^n + 0.5\psi(r_j)(C_i^n - C_{i-1}^n), \tag{4}$$