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# A shape calculus based method for a transmission problem with a random interface[☆]

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ABSTRACT

The present work is devoted to an approximation of the statistical moments of the solution of a class of elliptic transmission problems in $\mathbb{R}^3$ with uncertainly located transmission interfaces. In this model, the diffusion coefficient has a jump discontinuity across the random transmission interface which models linear diffusion in two different media separated by an uncertain surface. We apply the shape calculus approach to approximate the solution perturbation by the so-called *shape derivative*. Correspondingly, statistical moments of the solution are approximated by the moments of the shape derivative. We characterize the shape derivative as a solution of a related homogeneous transmission problem with nonzero jump conditions, which is solved by the boundary integral equation method. A rigorous theoretical framework is developed, and the theoretical findings are supported by and illustrated in two particular examples.



## 1. Introduction

Elliptic transmission or interface problems arise in many fields in science and engineering, such as tomography, deformation of an elastic body with inclusions, stationary groundwater flow in heterogeneous medium, fluid–structure interaction, scattering of an elastic body and many others. Combined with the state-of-the-art hardware, advanced numerical schemes are capable of producing a highly accurate and efficient deterministic numerical simulation, provided that the problem data are known exactly.

However, in real applications, a complete knowledge of the problem parameters is not realistic for many reasons. First, the simulation parameters are often estimated from measurements which can be *inexact*, e.g. due to imperfect measurement devices. Second, the parameters are estimated based on a large but *finite* number of system samples (snapshots); this information can be incomplete or stochastic. Finally, parameters of the system originate from a mathematical model which is itself only an approximation of the actual process. Under such circumstances, *highly accurate results of a single deterministic simulation for one particular set of problem parameters are of limited use*. An important paradigm, becoming rapidly popular

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over the past years, see e.g. [1–11] and the references therein, is to treat the lack of knowledge via modelling uncertain parameters as random fields.

If the forward solution operator is continuous, the solution of the forward problem with random parameters becomes a well-defined random field (since a composition of a continuous and a measurable function is measurable). Efficient numerical approximation of the random (or stochastic) solution and its probabilistic characteristics, e.g. statistical moments, is a highly non-trivial task representing numerous new interdisciplinary challenges: from regularity analysis and numerical analysis to modelling and efficient parallel large scale computing.

As already mentioned in the beginning, the case of uncertain but sufficiently regular interfaces appears in numerous applications, i.e. in microbiology and medicine. Interior domains may represent the interior of biological cells (or, say, a volume occupied by a human organ, tumour, etc.) and exterior domains the outer medium (the surrounding part of the human body). Cells of the same type (as well as human organs) have a similar shape which, however, is somewhat different between two particular species, so that the family of all possible shapes can be viewed as a perturbation of a nominal one. The interface between the interior and the exterior domain can be reasonably understood as the location of the jump of the material parameters, e.g. as the jump of the diffusivity.

In this article we develop a deterministic method for numerical solution for a class of transmission problems with randomly perturbed interfaces. The equation to be solved is of the form

$$-\nabla \cdot (\alpha \nabla u) = f \quad \text{in } D_{\pm},$$

where $D_-$ is a random bounded domain in $\mathbb{R}^3$ and $D_+ = \mathbb{R}^3 \setminus \overline{D_-}$ is its complement. The domains share a common random surface $\Gamma$, and the coefficient function $\alpha$ takes distinct constant values in $D_-$ and $D_+$, respectively. The solution $u$ is subject to jump conditions across $\Gamma$. A precise description of the model problem is deferred until Section 2.3, where a probabilistic perturbation model for the surface $\Gamma$ (and thus $D_{\pm}$) will be rigorously introduced. Within this model, the transmission interface depends on the "random event" $\omega$ and the parameter $\epsilon \geq 0$ controlling the amplitude of the perturbation. Therefore, the solution $u$ depends on $\omega$ and $\epsilon$, and will be denoted by $u^{\epsilon}(\omega)$. The case $\epsilon = 0$ corresponds to the *zero perturbation*. In the present paper we aim at estimating probabilistic properties of the solution perturbation $u^{\epsilon}(\omega) - u^0$ when the perturbation parameter is small, namely $\epsilon \ll 1$.

More precisely, we exploit the ideas from the recent publications [3,9,12–14] and propose to approximate the statistical moments of the solution perturbation by the moments of the linearized solution, i.e. for a fixed (small) value of the perturbation parameter $\epsilon$, the $k$th order statistical moments of the solution perturbation are approximated by

$$\mathcal{M}^k[u^{\epsilon} - u^0] \approx \epsilon^k \mathcal{M}^k[u'] \tag{1.1}$$

and similarly

$$\mathcal{M}^k[u^{\epsilon} - \mathbb{E}[u^{\epsilon}]] \approx \epsilon^k \mathcal{M}^k[u']. \tag{1.2}$$

Here $u'$ is the *shape derivative* of $u^{\epsilon}$ *formally* understood as the linear order term in the asymptotic expansion

$$u^{\epsilon}(\boldsymbol{x}, \omega) = u^0(\boldsymbol{x}) + \epsilon u'(\boldsymbol{x}, \omega) + \cdots, \quad \epsilon \to 0, \tag{1.3}$$

for almost all random events $\omega \in \Omega$ at a certain *fixed* point $\boldsymbol{x}$ in the Euclidean space $\mathbb{R}^3$. The notion of the shape derivative has been introduced in the context of the shape optimization (see e.g. the monograph [15] and the references therein) and allows to quantify sensitivity of the solution of a PDE to small perturbations of the boundary. It is worth mentioning that similar concepts have been developed (in the deterministic framework) and termed *domain derivatives* within the inverse problem community, see e.g. [16,17] for treatment of related elliptic and parabolic transmission problems in the case of the bounded outer subdomain $D_+$.

Although very intuitive, (1.3) cannot be used as a rigorous definition of $u'(\boldsymbol{x}, \omega)$. In particular, the existence of the shape derivative and the convergence of the asymptotic expansion (1.3) are unclear without further assumptions. In the first part of this article (Section 3) we develop a rigorous mathematical theory of existence of the shape derivative for the class of elliptic transmission problems under consideration. Similarly to [14, Lemma 1], we obtain a characterization of the shape derivative $u'(\boldsymbol{x}, \omega)$ as a solution of a deterministic transmission problem on a fixed interface. Our contribution in this section is two-fold: (i) we extend the notion of shape derivatives for interface problems developed in [14] to the case of unbounded domains and higher order moments, and (ii) we fill the gaps in the existing literature where only a limited rigorous discussion on existence of *shape derivatives* is presented. We point out the rigorous results in the deterministic setting in [16,17] and also in [18,19] where a similar concept of the Fréchet domain derivative has been presented and rigorously analysed. This derivative in fact coincides with the notion of the material derivative rather than the shape derivative in the terminology of [15] and Definition 3.5 below. We also refer to the related work [20] and references therein.

As mentioned above, for almost all $\omega \in \Omega$ the shape derivative $u'(\cdot, \omega)$ is a solution of a deterministic problem in $\mathbb{R}^3$ with (in general) nonhomogeneous jump conditions but with vanishing volume source term. The second contribution of this article is the derivation and analysis of boundary integral equation methods [21–23] which are used to solve this transmission problem on deterministic domains with deterministic interfaces. A tensorization argument is then used to obtain the approximation (1.1) for the statistical moments.