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# Isogeometric analysis with geometrically continuous functions on two-patch geometries

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ABSTRACT

We study the linear space of $C^s$-smooth isogeometric functions defined on a multi-patch domain $\Omega \subset \mathbb{R}^2$. We show that the construction of these functions is closely related to the concept of geometric continuity of surfaces, which has originated in geometric design. More precisely, the $C^s$-smoothness of isogeometric functions is found to be equivalent to geometric smoothness of the same order ($G^s$-smoothness) of their graph surfaces. This motivates us to call them $C^s$-smooth geometrically continuous isogeometric functions. We present a general framework to construct a basis and explore potential applications in isogeometric analysis. The space of $C^1$-smooth geometrically continuous isogeometric functions on bilinearly parameterized two-patch domains is analyzed in more detail. Numerical experiments with bicubic and biquartic functions for performing $L^2$ approximation and for solving Poisson's equation and the biharmonic equation on two-patch geometries are presented and indicate optimal rates of convergence.

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## 1. Introduction

In the framework of Isogeometric Analysis (IgA), which was introduced in [1], partial differential equations are discretized by using functions that are obtained from a parameterization of the computational domain. Typically one considers parameterizations by polynomial or rational spline functions (NURBS—non-uniform rational spline functions, see [2]) but other types of functions have been used also. On the one hand, this approach facilitates the data exchange with geometric design tools, since the mathematical technology used in Computer Aided Design (CAD) is based on parametric representations of curves and surfaces. On the other hand, it has been observed that the increased smoothness of the spline functions compared to traditional finite elements has a beneficial effect on stability and convergence properties [3,4].

Clearly, regular single-patch NURBS parameterizations are available only for domains that are topologically equivalent to a box. Though it is possible to extend the applicability of such parameterizations slightly by considering parameterizations with singular points (cf. [5]), it is preferable to use other techniques, due to the difficulties introduced by the use of singularities.

One of the most promising approaches is to use multi-patch parameterizations, which are coupled across their interfaces. Several coupling techniques are available, such as the direct identification of the degrees of freedom along the boundaries as in [6], the use of Lagrangian multipliers as in [7], or Nitsche's method [8]. The approximation power of T-spline representations, which are a generalization of NURBS that allow T-junctions and extraordinary vertices in the mesh (cf. [9]), was explored for two-patch geometries in [10]. However, these multi-patch constructions in isogeometric analysis are

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limited to functions of low regularity (at most $C^0$-smoothness). Consequently, the resulting numerical solutions are highly smooth almost everywhere, except across the interfaces between the patches of the multi-patch discretization.

Another approach is the use of trimmed NURBS geometries, which can also be combined with the multi-patch method. Such geometries have been used in the context of IgA (see e.g. [11–13]). However, trimming implies unavoidable gaps, when two trimmed NURBS patches are joined together (cf. [14]), and often requires advanced techniques for coupling the discretizations, see [12]. Another related technique is the use of mapped B-splines on general meshes [15].

The use of functions generated by subdivision algorithms has become a valuable alternative to NURBS, especially in Computer Graphics, since these functions lead to gap-free surfaces of arbitrary topology (cf. [16]). One of the standard subdivision methods is the Catmull–Clark subdivision, which generates surfaces consisting of bicubic patches, joined with $C^2$-smoothness everywhere except at extraordinary vertices, where they have a well-defined tangent plane. A Catmull–Clark based isogeometric method for solids is presented in [17]. Disadvantages of using subdivision methods are the possible reduction of the approximation power in the vicinity of extraordinary vertices, cf. [18] and the need for special numerical integration techniques. In fact these functions are piecewise polynomial functions with an infinite number of segments.

Another possibility to deal with domains of general topology is the use of T-splines, which can represent more complex geometries. This has been exploited in IgA, see e.g. [10,19]. However, the mathematical properties of the resulting isogeometric functions around the extraordinary vertices are not well understood. Around extraordinary vertices, T-splines are based on a special construction for geometrically continuous surfaces.

Geometric continuity is a well-known and highly useful concept in geometric design [20] and there exist numerous constructions for multi-patch surfaces with this property. It can be used to construct isogeometric functions of higher smoothness [21,22], but the systematic exploration of the potential for IgA has just started. Numerical experiments with a multi-patch parameterization of a disk have been presented in [18]. The results indicate again a reduction of the approximation power (and consequently a lower order of convergence) which is caused by the extraordinary vertices, similar to the case of subdivision algorithms.

Our paper consists of three main parts. Firstly we describe the concept of $C^s$-smooth geometrically continuous isogeometric functions on general multi-patch domains, and we present a general framework for computing a basis of the corresponding isogeometric discretization space in Section 2.

We then analyze the case of $C^1$-smooth geometrically continuous functions on bilinearly parameterized two-patch domains in Section 3. The dimension of the space of these isogeometric functions is investigated and a particular selection of the basis is proposed. In addition, generalizations of our approach to more general two-patch domains are discussed.

Finally, in order to demonstrate the potential of geometric continuity for IgA, we present numerical experiments to explore the approximation power of $C^1$-smooth geometrically continuous isogeometric functions for bilinearly parameterized two-patch geometries in Section 4. In addition to $L^2$ approximation and solving Poisson's equation, we also present results concerning the biharmonic equation, where the use of $C^1$-smooth test functions greatly facilitates the (isogeometric) discretization. Our numerical results indicate that the geometrically continuous representations maintain the full approximation power. This may be due to the fact that the effect of geometric continuity in our approach is not restricted to the vicinity of an extraordinary vertex as in earlier approaches, but spread out along the entire interface between the patches.

## 2. Geometrically continuous isogeometric functions

We present the concept of geometrically continuous isogeometric functions on general multi-patch domains. We show that geometric continuity of graphs of isogeometric functions is equivalent to standard continuity of isogeometric functions. Furthermore, we present a general framework for computing a basis of the corresponding isogeometric space.

### 2.1. $C^s$-smooth isogeometric functions

In order to simplify the presentation we restrict ourselves to the case of two-dimensional computational domains. Given a positive integer $n$, we consider $n$ bijective, regular geometry mappings

$$\boldsymbol{G}^{(\ell)} : [0,1]^2 \to \mathbb{R}^2, \quad \ell \in \{1,\ldots,n\},$$

which are represented in coordinates by

$$\boldsymbol{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \xi_2^{(\ell)}) \mapsto (G_1^{(\ell)}, G_2^{(\ell)}) = \boldsymbol{G}^{(\ell)}(\boldsymbol{\xi}^{(\ell)}),$$

with $\boldsymbol{G}^{(\ell)} \in \mathcal{S}^{(\ell)} \times \mathcal{S}^{(\ell)}$, where $\mathcal{S}^{(\ell)}$ is a tensor-product NURBS space of degree $d_\ell \in \mathbb{N}_0^2$. Consequently, each geometry mapping $\boldsymbol{G}^{(\ell)}$, $\ell \in \{1,\ldots,n\}$, is defined as a linear combination of NURBS basis functions $\psi_{\boldsymbol{i}}^{(\ell)} : [0,1]^2 \to \mathbb{R}$, i.e.,

$$\boldsymbol{G}^{(\ell)}(\boldsymbol{\xi}^{(\ell)}) = \sum_{\boldsymbol{i} \in I_\ell} \boldsymbol{d}_{\boldsymbol{i}}^{(\ell)} \psi_{\boldsymbol{i}}^{(\ell)}(\boldsymbol{\xi}^{(\ell)}),$$

with a suitable index set $I_\ell$ (a box in index space) and control points $\boldsymbol{d}_{\boldsymbol{i}}^{(\ell)} \in \mathbb{R}^2$. Thus it is a two-dimensional regular NURBS surface patch in $\mathbb{R}^2$. More precisely, we even assume that the geometry mappings $\boldsymbol{G}^{(\ell)}$ are defined and regular on a neighborhood of $[0,1]^2$.

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