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Simple a posteriori error estimators in adaptive isogeometric analysis

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ABSTRACT

In this article we propose two simple a posteriori error estimators for solving second order elliptic problems using adaptive isogeometric analysis. The idea is based on a *Serendipity*[1] *pairing* of discrete approximation spaces $S_h^{p,k}(\mathcal{M})$–$S_h^{p+1,k+1}(\mathcal{M})$, where the space $S_h^{p+1,k+1}(\mathcal{M})$ is considered as an enrichment of the original basis of $S_h^{p,k}(\mathcal{M})$ by means of the k-refinement, a typical unique feature available in isogeometric analysis. The space $S_h^{p+1,k+1}(\mathcal{M})$ is used to obtain a higher order accurate isogeometric finite element approximation and using this approximation we propose two simple a posteriori error estimators. The proposed a posteriori error based adaptive h-refinement methodology using LR B-splines is tested on classical elliptic benchmark problems. The numerical tests illustrate the optimal convergence rates obtained for the unknown, as well as the effectiveness of the proposed error estimators.

1. Introduction

1.1. Background

Isogeometric analysis (IGA) has been introduced in [1] as an innovative numerical methodology for the discretization of Partial Differential Equations (PDEs), the main idea was to improve the interoperability between CAD and PDE solvers, and to achieve this authors in [1] proposed to use CAD mathematical primitives, i.e. splines and NURBS, also to represent PDE unknowns. Isogeometric methods have been used and tested on a variety of problems of engineering interests, see [2,1] and references therein. The development on mathematical front start with h-approximation properties of NURBS in [3], and further studies for hpk-refinements in [4] and for anisotropic approximation in [5]. The recently published article in Acta Numerica [6] provides a complete overview in this direction. Non-uniform rational B-splines (NURBS) are the dominant geometric representation format for CAD. The construction of NURBS is based on a tensor product structure and, as a consequence, knot insertion is a global operation. To remedy this a local refinement can be achieved by breaking the global tensor product structure of multivariate splines and NURBS. In the current literature there are three different ways to achieve local refinements: T-splines [7–10], LR splines [11–13] and hierarchical splines [14,15,10,16–18]. Recently, there has been

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1 According to Wikipedia: *Serendipity means a "fortunate happenstance" or "pleasant surprise". It was coined by Horace Walpole in 1754. One aspect of Walpole's original definition of Serendipity is the need for an individual to be "sagacious" enough to link together apparently innocuous facts in order to come to a valuable conclusion*. We feel that this applies for the present discovery, but it is of course up to the readers to judge.

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much progress on the topic of the generalization of splines construction which allows for local refinement but an automatic reliable and efficient adaptive refinement routine is still one of the key issues in isogeometric analysis. To achieve a fully automatic refinement routine to solve PDEs problem in adaptive isogeometric analysis the *a posteriori error estimator* is required. This is the subject of current work.

The use of a posteriori error estimator in isogeometric analysis is still in its infancy. To the best of our knowledge only few work has been done in this direction, see [19,9,20–22,18,23–25]. The authors in [9] used the idea of hierarchical bases with bubble functions approach of Bank and Smith [26] to design a posteriori error estimator for T-splines, which was also considered in [19,18]. Another simple idea of explicit residual based error estimator has been explored in [27,28,22–25]. They require the computation of constants in Clement-type interpolation operators. Such constants are mesh (element) dependent and often incomputable for general element shape. A global constant can overestimate the local constants, and thus the exact error. Recently, a functional-type a posteriori error estimate for isogeometric discretization is presented in [20]. This type of error estimate, which was originally introduced in [29,30] on functional grounds (including integral identity and functional analysis arguments) is applicable for any conforming and non-conforming discretizations and known to provide a guaranteed and computable error bounds. But the hindrance in their popularity is due to high cost of computations which are based on solving a global minimization problem (Majorant minimization problem) in H(div) spaces. In [20], authors made an attempt to reduce the cost of computations for tensorial spline spaces but the same idea of cost reduction needs further study in adaptive isogeometric analysis. To the best of authors knowledge, in the above mentioned work on the use of a posteriori error estimators in isogeometric analysis the role of error estimator has been limited to either just as an indicator to perform adaptive refinement steps or the error estimation computation is given on tensorial mesh. A complete study about the performance of error estimators in adaptive analysis which makes them a suitable candidate for both the error estimation and adaptivity has not been considered so far. Recently, the present authors have presented a recovery based approach for establishing efficient error estimator in adaptive isogeometric analysis [31]. The approach is based on Superconvergent Patch Recovery (SPR) procedure (original idea of Zienkiewicz–Zhu [32]) that is enhanced to be applicable within isogeometric analysis. The enhancement includes the procedure for numerically computing the location of true superconvergent points. Extensive numerical tests have been performed on elliptic benchmark problems to show the efficiency of the developed SPR approach.

In this article we present another possibility to design a posteriori error estimators in adaptive isogeometric analysis. The employed technique is based on solving the original problem with two discretization schemes of different accuracy and using the difference in the approximations as an estimate of the error, see [33] and Chapter 5 in [34]. Consider the elliptic model problems of Section 5.1 and suppose that the numerical approximation u_h in Finite Element (FE) subspace V_h is known. Then in classical Finite Element Methods (FEM), the enhanced space V_h^* may, for example, be constructed by either global h-refinement or p-refinement of the mesh used to construct the original FE subspace V_h, see [34,35,26,36,37]. Suppose $u_h^* \in V_h^*$ is the another FE approximation to the original problem then after using the triangle inequality on the energy error (the energy norm is induced by the bilinear form of the underlying self adjoint elliptic problem as given by Eq. (35)) can be written as

$$\|e\|_E = \|u-u_h\|_E \le \underbrace{\|u_h^*-u_h\|_E}_{\text{Computable}} + \underbrace{\|u-u_h^*\|_E}_{\text{Non-Computable}} . \tag{1}$$

If we assume that the approximation $u_h^* \in V_h^*$ is superior to the original approximation u_h, then

$$\|e\|_E \approx \|u_h^*-u_h\|_E = \eta_h^* \quad \text{(Computable error estimate)}. \tag{2}$$

The enhanced subspace V_h^* based on global h- or p-refinement of the element of original subspace V_h clearly satisfies $V_h \subset V_h^*$. From a priori error estimation results in classical FEM, for a sufficiently smooth solution u it has been observed that $\|u-u_h^*\|_E \le C_\theta \|u-u_h\|_E$, where $C_\theta \in [0,1)$ for h-refined subspace V_h^* and $C_\theta = O(h)$ for p-refined subspace V_h^*. It is seen in the literature that the adaptive simulations based on the error estimator η_h^* also provide the asymptotic exactness result on refined meshes, see [34,35,26,36,37]. The attractiveness of such ideas stems from their applicability to quite general classes of problems combined with simplicity and ease of implementation.

In isogeometric analysis, there are several possibilities to obtain a higher order approximation u_h^* from the space V_h^*. In comparison to the h- and p-refinements available in classical FEA, isogeometric analysis offers a new possibility of k-refinement in which the global continuity and degree are increased together. Suppose $S_h^{p,k}(\mathcal{M})$ is the given isogeometric FE subspace of degree p, continuity k with size of elements h on the mesh $\mathcal{M}$. Then the following approximation spaces can be obtained under these operations:

$$S_h^{p,k}(\mathcal{M}) \xrightarrow{h\text{-refinement}} S_{h/2}^{p,k}(\bar{\mathcal{M}}) \tag{3}$$

$$S_h^{p,k}(\mathcal{M}) \xrightarrow{p\text{-refinement}} S_h^{p+1,k}(\mathcal{M}) \tag{4}$$

$$S_h^{p,k}(\mathcal{M}) \xrightarrow{k\text{-refinement}} S_h^{p+1,k+1}(\mathcal{M}) \tag{5}$$