Computers and Mathematics with Applications 70 (2015) 1583–1605

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Optimal additive Schwarz methods for the *hp*-BEM: The hypersingular integral operator in 3D on locally refined meshes

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ARTICLE INFO

Article history:
Available online 6 August 2015

Keywords:
hp-BEM
Hypersingular integral equation
Preconditioning
Additive Schwarz method

ABSTRACT

We propose and analyze an overlapping Schwarz preconditioner for the *p* and *hp* boundary element method for the hypersingular integral equation in 3D. We consider surface triangulations consisting of triangles. The condition number is bounded uniformly in the mesh size *h* and the polynomial order *p*. The preconditioner handles adaptively refined meshes and is based on a local multilevel preconditioner for the lowest order space. Numerical experiments on different geometries illustrate its robustness.

1. Introduction

Many elliptic boundary value problems that are solved in practice are linear and have constant (or at least piecewise constant) coefficients. In this setting, the boundary element method (BEM, [1–4]) has established itself as an effective alternative to the finite element method (FEM). Just as in the FEM applied to this particular problem class, high order methods are very attractive since they can produce rapidly convergent schemes on suitably chosen adaptive meshes. The discretization leads to large systems of equations, and a use of iterative solvers brings the question of preconditioning to the fore.

In the present work, we study high order Galerkin discretizations of the hypersingular operator. This is an operator of order 1, and we therefore have to expect the condition number of the system matrix to increase as the mesh size h decreases and the approximation order p increases. We present an additive overlapping Schwarz preconditioner that offsets this degradation and results in condition numbers that are bounded independently of the mesh size and the approximation order. This is achieved by combining the recent $H^{1/2}$-stable decomposition of spaces of piecewise polynomials of degree p of [5] and the multilevel diagonal scaling preconditioner of [6,7] for the hypersingular operator discretized by piecewise linears.

Our additive Schwarz preconditioner is based on stably decomposing the approximation space of piecewise polynomials into the lowest order space (i.e., piecewise linears) and spaces of higher order polynomials supported by the vertex patches. Such stable localization procedures were first developed for the *hp*-FEM in [8] for meshes consisting of quadrilaterals (or, more generally, tensor product elements). The restriction to tensor product elements stems from the fact that the localization is achieved by exploiting stability properties of the 1D-Gauß–Lobatto interpolation operator, which, when applied to polynomials, is simultaneously stable in L^2 and H^1 (see, e.g., [9, eqns. (13.27), (13.28)]). This simultaneous stability raises the hope for $H^{1/2}$-stable localizations and was pioneered in [10] for the *hp*-BEM for the hypersingular operator on meshes

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http://dx.doi.org/10.1016/j.camwa.2015.06.025
0898-1221/

consisting of quadrilaterals. Returning to the *hp*-FEM, H^1-stable localizations on triangular/tetrahedral meshes were not developed until [11]. The techniques developed there were subsequently used in [5] to design $H^{1/2}$-stable decompositions on triangular meshes and thus paved the way for condition number estimates that are uniform in the approximation p for overlapping Schwarz methods for the *hp*-version BEM applied to the hypersingular operator. Non-overlapping additive Schwarz preconditioners for high order discretizations of the hypersingular operator are also available in the literature, [12]; as it is typical of this class of preconditioners, the condition number still grows polylogarithmically in p.

Our preconditioner is based on decomposing the approximation space into the space of piecewise linears and spaces associated with the vertex patches. It is highly desirable to decompose the space of piecewise linears further in a multilevel fashion. For sequences of uniformly refined meshes, the first such multilevel space decomposition appears to be [13] (see also [14]). For adaptive meshes, local multilevel diagonal scaling was first analyzed in [15], where for a sequence $\mathcal{T}_\ell$ of successively refined adaptive meshes a uniformly bounded condition number for the preconditioned system is established. Formally, however, [15] requires that $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1} \subset \mathcal{T}_{\ell+k}$ for all $\ell, k \in \mathbb{N}_0$, i.e., as soon as an element $K \in \mathcal{T}_\ell$ is not refined, it remains non-refined in all succeeding triangulations. While this can be achieved implementationally, the recent works [6,7] avoid such a restriction by considering sequences of meshes that are obtained in typical h-adaptive environments with the aid of *newest vertex bisection* (NVB). We finally note that the additive Schwarz decomposition on adaptively refined meshes is a subtle issue. Hierarchical basis preconditioners (which are based on the new nodes only) lead to a growth of the condition number with $\mathcal{O}\left(|\log h_{\min}|^2\right)$; see [16]. Global multilevel diagonal preconditioning (which is based on all nodes) leads to a growth $\mathcal{O}\left(|\log h_{\min}|\right)$; see [17,6].

The paper is organized as follows: In Section 2 we introduce the hypersingular equation and the discretization by high order piecewise polynomial spaces. Section 3 collects properties of the fractional Sobolev spaces including the scaling properties. Section 4 studies in detail the p-dependence of the condition number of the unpreconditioned system. The polynomial basis on the reference triangle chosen by us is a hierarchical basis of the form first proposed by Karniadakis and Sherwin, [18, Appendix D.1.1.2]; the precise form is the one from [19, Section 5.2.3]. We prove bounds for the condition number of the stiffness matrix not only in the $H^{1/2}$-norm but also in the norms of L^2 and H^1. This is also of interest for *hp*-FEM and could not be found in the literature. Section 5 develops several preconditioners. The first one (Theorem 5.3) is based on decomposing the high order approximation space into the global space of piecewise linears and local high order spaces of functions associated with the vertex patches. The second one (Theorem 5.6) is based on a further multilevel decomposition of the global space of piecewise linears. The third one (Theorem 5.10) exploits the observation that topologically, only a finite number of vertex patches can occur. Hence, significant memory savings for the preconditioner are possible if the exact bilinear forms for the vertex patches are replaced with scaled versions of simpler ones defined on a finite number of reference configurations. Numerical experiments in Section 6 illustrate that the proposed preconditioners are indeed robust with respect to both h and p.

We close with a remark on notation: The expression $a \lesssim b$ signifies the existence of a constant $C > 0$ such that $a \le C\, b$. The constant C does not depend on the mesh size h and the approximation order p, but may depend on the geometry and the shape regularity of the triangulation. We also write $a \sim b$ to abbreviate $a \lesssim b \lesssim a$.

2. *hp*-discretization of the hypersingular integral equation

2.1. Hypersingular integral equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedron with a connected boundary $\partial\Omega$, and let $\Gamma \subseteq \partial\Omega$ be an open, connected subset of $\partial\Omega$. If $\Gamma \neq \partial\Omega$, we assume it to be a Lipschitz hypograph, [4]; the key property needed is that Γ is such that the ellipticity condition (2.2) holds. Furthermore, we will use affine, shape regular triangulations of Γ, which further imposes conditions on Γ. In this work, we are concerned with preconditioning high order discretizations of the hypersingular integral operator, which is given by

$$(Du)\,(x) := -\partial_{n_x}^{int} \int_\Gamma \partial_{n_y}^{int} G(x,y)u(y)\, ds_y \quad \text{for } x \in \Gamma, \tag{2.1}$$

where $G(x,y) := \frac{1}{4\pi}\frac{1}{|x-y|}$ is the fundamental solution of the 3D-Laplacian and $\partial_{n_y}^{int}$ denotes the (interior) normal derivative with respect to $y \in \Gamma$.

We will need some results from the theory of Sobolev and interpolation spaces, see [4, Appendix B]. For an open subset $\omega \subset \partial\Omega$, let $L^2(\omega)$ and $H^1(\omega)$ denote the usual Sobolev spaces. The space $\widetilde{H}^1(\omega)$ consists of those functions whose zero extension to $\partial\Omega$ is in $H^1(\partial\Omega)$. (In particular, for $\omega = \partial\Omega$, $H^1(\partial\Omega) = \widetilde{H}^1(\partial\Omega)$.) When the surface measure of the set $\partial\Omega \setminus \omega$ is positive, we use the equivalent norm $\|u\|^2_{\widetilde{H}^1(\omega)} := \|\nabla_\Gamma u\|^2_{L^2(\omega)}$.

We will define fractional Sobolev norms by interpolation. Proposition 2.1 collects key properties of interpolation spaces that we will need; we refer to [20,21] for a comprehensive treatment. For two Banach spaces $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$, with continuous inclusion $X_1 \subseteq X_0$ and a parameter $s \in (0,1)$ the interpolation norm is defined as

$$\|u\|^2_{[X_0,X_1]_s} := \int_{t=0}^\infty t^{-2s}\left(\inf_{v\in X_1} \|u-v\|_0 + t\|v\|_1\right)^2 \frac{dt}{t}.$$

The interpolation space is given by $[X_0, X_1]_s := \left\{u \in X_0 : \|u\|_{[X_0,X_1]_s} < \infty\right\}$.