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Mesh grading in isogeometric analysis

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ABSTRACT

This paper is concerned with the construction of graded meshes for approximating so-called singular solutions of elliptic boundary value problems by means of multipatch discontinuous Galerkin Isogeometric Analysis schemes. Such solutions appear, for instance, in domains with re-entrant corners on the boundary of the computational domain, in problems with changing boundary conditions, in interface problems, or in problems with singular source terms. Making use of the analytic behavior of the solution, we construct the graded meshes in the neighborhoods of such singular points following a multipatch approach. We prove that appropriately graded meshes lead to the same convergence rates as in the case of smooth solutions with approximately the same number of degrees of freedom. Representative numerical examples are studied in order to confirm the theoretical convergence rates and to demonstrate the efficiency of the mesh grading technology in Isogeometric Analysis.

1. Introduction

The gradient of the solution of elliptic boundary value problems can exhibit singularities in the vicinities of re-entrant corners or edges. The same is true in case of changing boundary conditions or interface problems. This singular behavior of the gradients was discovered and analyzed in the famous work by Kondrat'ev [1]. We refer the reader to the monographs [2–4] for a more recent and comprehensive presentation of related results. It is well known that these singularities may cause loss in the approximation order of the standard discretization methods like the finite element method, see the classical monograph [5] or the more recent paper [6]. In the case of two dimensional problems with singular boundary points, grading mesh techniques have been developed for finite element methods in order to recover the full approximation order, see the classical textbook [7] and the more recent publications [6,8–10] for three-dimensional problems. Here, we devise graded meshes for solving elliptic problems with singular solutions by means of discontinuous Galerkin Isogeometric Analysis method (dG IgA).

In the frame of IgA, the use of B-splines or NURBS basis functions allows complicated CAD (Compute-Aided Design) geometries to be exactly represented. The key point of Hughes et al. [11] was to make use of the same basis to approximate the solution of the problem under consideration. Since this pioneer paper, applications of the IgA approach have been considered in many fields, see [12]. Here, we apply a multipatch symmetric dG IgA method which has been extensively studied for diffusion problems in volumetric computational domains and on surfaces in [13,14], respectively, see also [15] for a comprehensive presentation. The solution of the problem is independently approximated in every subdomain by IgA without imposing any matching grid conditions and without any continuity requirements for the discrete solution across

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the subdomain interfaces. Symmetrized numerical fluxes with interior penalty jump terms, see, e.g., [16–18], are introduced on the interfaces in order to treat the discontinuities of the discrete solution and to interchange information between the non-matching grids. As we will see later, considering a numerical scheme in this general context is a flexible approach, notably when applied on zone-type subdivisions of Ω, which are convenient for analysing elliptic boundary value problems in domains with singular boundary points.

This paper aims at the construction of graded dG IgA meshes in the zones located near the singular points in order to recover full convergence rates like in the case of smooth solutions on uniform meshes. The grading of the mesh is mainly determined by the analytic behavior of the solution u around the singular points and follows the spirit of grading mesh techniques using layers, which have been proposed for finite element methods in [7,6,8]. According to this, having an a priori knowledge about the location of the singular point, e.g. the re-entrant corner, the domain Ω is subdivided into zones, called layers in [6,8], and then a further subdivision of Ω into subdomains (also called patches in IgA), say $\mathcal{T}_H(\Omega) := \{\Omega_i\}_{i=1}^N$, is performed in such a way that $\mathcal{T}_H(\Omega)$ is in correspondence with the initial zone partition. On the other hand, the solution can be split into a sum of a regular part $u_r \in W^{l\geq 2,2}(\Omega)$ and a singular part $u_s \in W^{1+\varepsilon,2}(\Omega)$, with known $\varepsilon \in (0, 1)$, i.e., $u = u_r + u_s$, see, e.g., [2]. The analytical form of u_s contains terms with singular exponents in the radial direction. We use this information and construct appropriately graded meshes in the zones around the singular points. The resulting graded meshes have a "zone-wise character", this means that the grid size of the graded mesh in every zone determines the mesh of every subdomain which belongs to this zone, where we assume that every subdomain belongs to only one zone (the ideal situation is every zone to be a subdomain). We mention that the mesh grading methodology is developed and is analyzed for the classical two dimensional problem with a re-entrant corner. The proposed methodology can be generalized and applied to other situations. This is shown by the numerical examples presented in Section 4.

The particular properties of the produced graded meshes help us show optimal error estimates for the dG IgA method, which exhibit optimal convergence rates. The error estimates for the proposed method are proved by using a variation of Céa's lemma and using B-spline quasi-interpolation estimates for $u \in W^{1,2}(\Omega) \cap W^{l\geq 2,p\in(1,2]}(\mathcal{T}_H(\Omega))$, which have been proved in [13]. More precisely, these interpolation estimates are expressed with respect to the mesh size h_i of the corresponding subdomain Ω_i. For the domains away from the singular point, the solution is smooth (see u_r part in previous splitting), and we can derive the usual interpolation estimates. Conversely, for the subdomains Ω_i, for which the boundary $\partial\Omega_i$ touches the singular point, the singular part u_s of the solution u can be considered as a function from the Sobolev space $W^{2,p\in(1,2)}(\Omega_i)$. Now the estimates given in [13] enable us to derive error estimates for the singular part u_s. This makes the whole error analysis easier in comparison with the techniques earlier developed for the finite element method, e.g., in [6,19,9].

We mention that, in the literature, other IgA techniques have been proposed for solving problems with singularities. In [20,21], the mapping technique has been developed, where the original B-spline finite dimensional space has been enriched by generating singular functions which resemble the types of the singularities of the problem. The mappings constructed on this enriched space describe the geometry singularities explicitly. Also in [22], by studying the anisotropic character of the singularities of the problem, the one-dimensional approximation properties of the B-splines are generalized for two-dimensional problems, in order to produce anisotropic refined meshes in the regions of the singular points.

Furthermore, we note the extended isogeometric element formulation, which has been developed for solving fracture mechanics problems in [23–26]. Utilizing knot insertion, the original NURBS basis is enriched by crack-discontinuous functions, which are able to reproduce the singular field near the crack. In [27–29], the flexibility of the PHT-spline and T-spline basis on local adaptive refinement has been used for solving elasticity problems on non-smooth domains. Recently, other types of adaptivity refinement procedures have been presented by using hierarchical B-splines, see [30]. The authors introduce different levels of resolution in an adaptive framework, keeping simultaneously the good properties of standard B-splines. The same methodology has been generalized for truncated hierarchical B-splines and has been applied in CAD modeling as well in [31]. Lastly, new adaptive refinement techniques have been developed in the IgA framework on the basis of functional a posteriori error estimates in [32].

The rest of the paper is organized as follows. The problem description, the weak formulation and the dG IgA discrete analogue are presented in Section 2. Section 3 discusses the construction of the appropriately graded IgA meshes, and provides the proof for obtaining the full approximation order of the dG IgA method on the graded meshes. Several two and three dimensional examples are presented in Section 4. Finally, we draw our conclusions.

2. Problem description and dG IgA discretization

First, let us introduce some notation. We define the *differential operator*

$$D^a = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad \text{with } D_j = \frac{\partial}{\partial x_j}, D^{(0,\ldots,0)}u = u, \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, with $\alpha_j \geq 0, j = 1, \ldots, d$, denotes a multi-index of the degree $|\alpha| = \sum_{j=1}^d \alpha_j$. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ we denote by $W^{l,p}(\Omega)$, with $l \geq 1$ and $1 \leq p \leq \infty$, the usual Sobolev function spaces