Computers and Mathematics with Applications 70 (2015) 47–65

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Nonconforming Least-Squares Spectral Element Method for European Options [☆]

Arbaz Khan [a,*], Pravir Dutt [a], Chandra Shekhar Upadhyay [b]

[a] Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, UP, 208016, India
[b] Department of Aerospace Engineering, Indian Institute of Technology Kanpur, UP, 208016, India

ARTICLE INFO

Article history:
Received 20 July 2014
Received in revised form 4 February 2015
Accepted 19 April 2015
Available online 12 May 2015

Keywords:
Black–Scholes equation
Hermite mollifier
Least-Squares method
Domain decomposition
Parallel preconditioners
Exponential accuracy

ABSTRACT

Several methods have been proposed in the literature for solving the Black–Scholes equation for European Options. The method proposed in the current study achieves spectral accuracy in both space and time. The method is based on minimization of a functional given in terms of the sum of squares of the residuals in the partial differential equation and initial condition in different Sobolev norms, and a term which measures the jump in the function and its derivatives across inter-element boundaries in appropriate fractional Sobolev norms. To obtain values of the solution and its derivatives the initial condition is mollified and the computed solution is post processed. Error estimates are obtained for this method. Specific numerical examples are given to show the efficiency of this method.

1. Introduction

Consider the Black–Scholes (BS) equation [1,2] for European Option

$$V_\tau + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \quad \text{in } (0,\infty)\times[0,T] \tag{1.1}$$

where V, S, r and σ are respectively Option price, underlying asset price, risk-free interest rate and volatility.

Now, we define the European Call Option and the European Put Option.

Definition 1.1. In European Call Option the holder has the right, but not the obligation, to buy an asset at a prescribed price K (strike price) at maturity time T in future. The payoff function for European Call Option is

$$V_C(S,T) = \max(S-K,0). \tag{1.2}$$

Definition 1.2. In European Put Option the holder has the right, but not the obligation, to sell an asset at a prescribed price K (strike price) at maturity time T in future. The payoff function for European Put Option is

$$V_P(S,T) = \max(K-S,0). \tag{1.3}$$

☆ Research supported by CSIR (Council of Scientific and Industrial Research, India) Grant 09/092(0712)/2009-EMR-I.

* Corresponding author.
E-mail addresses: arbaz@iitk.ac.in, arbazkha@gmail.com (A. Khan), pravir@iitk.ac.in (P. Dutt), shekhar@iitk.ac.in (C.S. Upadhyay).

http://dx.doi.org/10.1016/j.camwa.2015.04.019
0898-1221/

Recently, Spectral methods [3] have been used to solve Option Pricing problems. In 2000, Bunin et al. [4] proposed Chebyshev Collocation methods to solve the European Call Option problem on parallel computers. After this, Greenberg [5] solved American Options problem by Chebyshev Tau method. For smooth initial conditions, De Frutos [6] has presented a Laguerre–Galerkin Spectral Method to price bonds. More recently, Zhu et al. [7] have used a Spectral element method to price European Options. These methods give quadratic accuracy in time, while being spectrally accurate in space. Schötzau et al. [8] proposed *hp*-version of the Discontinuous Galerkin Finite Element Method to solve parabolic problems. In [9], Dutt et al. proposed Least-Squares Spectral Element Method for parabolic partial differential equations (PDE) on bounded domains and proved exponential accuracy for analytic data.

In this paper, we develop a Non-Conforming Least-Squares Spectral Element Method (LSSEM) for parabolic initial value problems with nonsmooth, unbounded initial data and variable coefficients on unbounded domains using parallel computers. One of the applications of this method is in finance, namely Black–Scholes equation for European Options. It will be shown that the proposed LSSEM is exponentially accurate in both space and time. Sobolev spaces of different orders in space and time are used for the results, as presented in [10]. If the data belong to certain Gevrey spaces then the solution also belongs to a Gevrey space [11].

The proposed method is a Least-Squares method as presented in [9]. The space domain is an interval which is divided into a number of sub-intervals. The functional is the sum of the squares of the residuals in the partial differential equation and initial condition in different Sobolev norms, and a term which measures the jump in the function and its derivatives across inter-element boundaries in appropriate fractional Sobolev norms. We minimize the functional on a given time interval. Hermite mollifiers, as described in [12,13], are used to resolve the difficulty of non-smooth initial data.

Now we describe the organization of this paper. In Section 2 the function spaces and a priori estimates for parabolic initial value problem, as presented in [14,10,11], are given. Discretization of the domain and stability estimates are discussed in Section 3. In Section 4 we describe the numerical scheme, parallelization and preconditioning for our method. Estimate for non smooth initial condition, in negative Sobolev norms, is presented in Section 5. In Section 6 error estimates are obtained for this method. Finally, in Section 7 specific numerical examples are provided to show the effectiveness of the method.

2. Function spaces

We consider $\Omega = \mathbb{R}$ as the domain of the logarithmic price $x = \log(S/K)$ and define $t = \frac{T-\tau}{T}$ on the time interval $I = [0, 1]$. We shall focus here on the Black–Scholes equation for the European call with the assumption that the rate of interest r and volatility σ are smooth (or even analytic) functions of x and t with bounded derivatives. The coefficients a, b and c belong to $\mathscr{D}_{2,1}(\Omega \times I)$ as defined in [11] and satisfy

$$\|D_x^i D_t^j a(x,t)\|_{L^\infty(\Omega\times I)} \le AB^{i+j}\; i!(j!)^2,$$

where A and B are positive numbers.

The price $u(x, t)$ has to satisfy the BS equation

$$\begin{aligned} \mathcal{L}u &= u_t - au_{xx} - bu_x - cu = 0 \quad \text{in } \Omega \times I, \\ u(x,0) &= f(x) \quad \text{in } \Omega \times \{0\}. \end{aligned} \tag{2.1}$$

Note that $f(x)$ may not be in $L^2(\Omega)$, for example

$$f(x) = (Ke^x - K)^+.$$

To resolve this difficulty, let us define

$$v(x,t) = u(x,t)\,\mathrm{sech}(\eta x), \tag{2.2}$$

where $\eta > 0$ is sufficiently large so that the initial data

$$v(x,0) = u(x,0)\ \mathrm{sech}(\eta x), \tag{2.3}$$

is such that $ve^{\mu x}, ve^{-\mu x} \in L^2(\Omega)$ for some $\mu > 0$.

Substituting $v(x, t)$ in Eq. (2.1), we get the partial differential equation that v satisfies, as:

$$\begin{aligned} \mathbb{L}v &= v_t - \alpha\ v_{xx} - \beta\ v_x - \gamma\ v = 0 \quad \text{in } \Omega \times I, \\ v(x,0) &= f(x)\ \ \mathrm{sech}(\eta x) = \ g(x) \quad \text{in } \Omega \times \{0\}. \end{aligned} \tag{2.4}$$

We assume the coefficients a, b and c in (2.1) are smooth or even analytic and all derivatives are bounded. Clearly the same assumption will continue to hold for the coefficients $\alpha = a$, $\beta = 2\ a\ \eta\ \tanh \eta x + b$ and $\gamma = \eta^2 a + b\eta \tanh \eta x + c$, since $\tanh \eta x$ has bounded derivatives of all orders. Moreover the coefficients belong to $\mathscr{D}_{2,1}(\bar{\Omega} \times [0, 1])$.

However, the initial data $g(x) = f(x)\ \mathrm{sech}\ \eta x$ is not smooth. To resolve this difficulty we use the Hermite mollifiers [12]:

$$\Phi(x) = e^{-\frac{x^2}{2}} \sum_{j=0}^{P} \frac{(-1)^j}{4^j j!} H_{2j}\left(\frac{x}{\sqrt{2}}\right). \tag{2.5}$$