Computers and Mathematics with Applications 69 (2015) 477–493

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Coupled nonlinear advection–diffusion–reaction system for prevention of groundwater contamination by modified upwind finite volume element method[☆]

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ARTICLE INFO

Article history:
Received 21 May 2014
Received in revised form 29 November 2014
Accepted 1 January 2015
Available online 14 February 2015

Keywords:
Finite volume element
Modified upwind approximation
Error estimates
Nonlinear problem
Two-grid method

ABSTRACT

The coupled advection-dominated diffusion reaction equations arising in the prevention of groundwater contamination problem are approximated by the modified upwind finite volume element method. In order to solve the resulting nonlinear system efficiently, we use a two-grid algorithm to decompose the nonlinear system into a small nonlinear system on a coarse grid with mesh size H and a linear system on a fine grid with mesh size h. It is shown that the approximation still achieves asymptotically optimal as long as the mesh sizes satisfy $H = O(h^{1/3})$. Numerical examples are presented to illustrate efficiency and accuracy of the proposed method.

1. Introduction

We shall consider a mathematical model for prevention and control of groundwater contaminant with the microbiological technology. The model can be described as the following coupled nonlinear advection–diffusion–reaction equations,

$$\begin{cases} S_t + \vartheta(\mathbf{x}) \cdot \nabla S - \nabla \cdot (D(\mathbf{x})\nabla S) + M_p e_1 f(S, A) = 0, & \text{in } \Omega \times (0, T], \\ A_t + \vartheta(\mathbf{x}) \cdot \nabla A - \nabla \cdot (D(\mathbf{x})\nabla A) + M_p e_2 f(S, A) = 0, & \text{in } \Omega \times (0, T], \\ (M_s)_t + \vartheta(\mathbf{x}) \cdot \nabla M_s - \nabla \cdot (D(\mathbf{x})\nabla M_s) + M_s e_3 f(S, A) + r(\mathbf{x}) M_s = 0, & \text{in } \Omega \times (0, T], \\ S\,|_{\partial\Omega} = A\,|_{\partial\Omega} = M_s\,|_{\partial\Omega} = 0, & \text{on } \partial\Omega \times (0, T], \\ S(\mathbf{x}, 0) = S_0(\mathbf{x}), \quad A(\mathbf{x}, 0) = A_0(\mathbf{x}), \quad M_s(\mathbf{x}, 0) = (M_s)_0(\mathbf{x}), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. S is the concentration of the main ground substance, A is the aqueous solution electrolyte concentration, and M_s is the concentration of microorganism (e.g. bacteria). The velocity vector $\vartheta(\mathbf{x}) = (\vartheta_1(\mathbf{x}), \vartheta_2(\mathbf{x}))$, which are given functions in Ω, stands for the average linearized groundwater velocity. $D(\mathbf{x})$ is a hydrodynamic diffusion function. M_p is the total concentration of active microorganism and $M_s = M_p/R_M$

☆ The work is supported by the National Natural Science Foundation of China Grant No. 11401289 and DR Fund of Ludong University No. LY2015005.
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http://dx.doi.org/10.1016/j.camwa.2015.01.001
0898-1221/

with a positive constant R_M. e_i are positive constants for $i = 1, 2, 3$. Here,

$$f(S, A) = \frac{S}{K_s + S} \cdot \frac{A}{K_a + A}$$

and $K_s,\ K_a$ are positive constants. We can refer to [1] about the meanings of the notations for details.

We suppose the variable coefficients $\vartheta(\mathbf{x})$, $D(\mathbf{x})$ and $r(\mathbf{x})$ satisfy

$$\begin{aligned}
&(1)\ 0 < k_1 \le D(\mathbf{x}) \le k_2, && D(\mathbf{x}) \in W^1_\infty(\Omega);\\
&(2)\ |\vartheta_i(\mathbf{x})| \le \alpha_i, && \vartheta_i(\mathbf{x}) \in W^1_\infty(\Omega);\\
&(3)\ r(\mathbf{x}) \le r_1;
\end{aligned}$$

where r_1 and k_i, α_i $(i = 1, 2)$ are positive constants.

Modified upwind finite volume method as an important numerical tool for solving the convection-dominated diffusion problems inherits the advantage of upwind finite volume scheme. It can achieve the unconditional stability and eliminate the excessive numerical dispersion effectively in the process of obtaining the approximation solutions. Besides, with superiority to the upwind scheme, the modified upwind method improves the accuracy to second order in space increment. As a result, it is popular in the mathematical models that arise in petroleum reservoir simulation, subsurface contaminant transport, seawater intrusion and many other significant problems. In the past several decades, many researchers have studied modified upwind finite volume method extensively and obtained some important results. We refer to [2–7] and the references cited therein.

On the other hand, the two-grid method, based on two conforming spaces, was first introduced by Xu [8,9] for the non-symmetric and nonlinear elliptic problems. The basic idea of this approach is to firstly solve the original nonlinear equations on a coarse grid with mesh size H and then solve a linearized problem on the fine grid with mesh size h by using Newton iteration once. Later on, the two-grid method was further investigated by many authors. For instance, Liu–Rui–Guo [10] gave the detailed analysis of this approach to nonlinear reaction–diffusion problem by expanded mixed finite element method, Bi–Ginting [11] used the finite volume element method with two-grid method to obtain the solutions of linear and nonlinear elliptic problems, Chen–Liu [12] applied two-grid finite volume element method to the semi-linear parabolic equations, Wu–Allen [13] and Chen–Chen [14] provided two-grid techniques to the reaction–diffusion equation, Dawson–Wheeler–Woodward [15] studied the two-grid finite difference scheme for parabolic problems.

In this paper, we apply modified upwind finite volume element scheme to approximate the coupled nonlinear problem (1.1) with the advection-dominated term. Unfortunately, the resulting algebraic system of equations is a large coupled system of nonlinear equations. So it is necessary for us to use the two-grid method. By this approach, solving the coupled nonlinear system on the fine space is reduced to solving a linear system on the fine space and a smaller nonlinear system on a coarse space. Furthermore, it is proved and illustrated that this method can obtain the numerical solutions without sacrificing the order of accuracy and achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{1/3})$. As we know, there is no modified upwind finite volume element convergence analysis for this kind of advection–diffusion–reaction equations by using two-grid techniques. Here, we present the method and error estimates that partly fill this gap.

The remainder of the paper is organized as follows. In Section 2, we present some notations and describe the finite volume element scheme for system (1.1). The modified upwind approximation formulation is introduced and error estimate in L^2 norm is presented in Section 3. In Section 4, the two-grid method is proposed and convergence results in H^1 norm are shown that the method has no loss in accuracy. Some numerical examples are used to testify the theoretical results in Section 5.

Throughout this paper, we use C to denote a generic positive constant independent of the discretization parameters with possibly different values in different contexts.

2. Notations and preliminaries

For the simple presentation, we assume the domain Ω is a rectangle with its sides parallel to the axes x_1 and x_2, we apply the grid for the cell-centered finite volume scheme, mainly because of their good conservation properties which are popular in fluid simulation, heat transfer and weather prediction problems. After covering the plane $\mathbf{R^2}$ by square cells with sides of length h, we denote the grid points by $P_{i,j} = (x_{1,i}, x_{2,j}) = (ih, jh)$, where $i, j = 0, 1, \ldots, M$ are integer indices. For the Dirichlet boundary condition, we shall suppose the $\partial\Omega$ passes through the grid points as follows:

$$0 = x_{1,0} < x_{1,\frac{1}{2}} < x_{1,1} < \cdots < x_{1,M-1} < x_{1,M-\frac{1}{2}} < x_{1,M} = 1;$$

and

$$0 = x_{2,0} < x_{2,\frac{1}{2}} < x_{2,1} < \cdots < x_{2,M-1} < x_{2,M-\frac{1}{2}} < x_{2,M} = 1,$$

where $x_{1,i\pm\frac{1}{2}} = x_{1,i} \pm \frac{h}{2},\ x_{2,j\pm\frac{1}{2}} = x_{2,j} \pm \frac{h}{2}$.

Let

$$\begin{aligned}
&\overline{\omega} = \{(x_{1,i}, x_{2,j}) \in \overline{\Omega} :\ i, j = 0, 1, \ldots, M\}, \qquad \omega = \overline{\omega} \cap \Omega, \qquad \Gamma = \overline{\omega} \setminus \omega,\\
&\Gamma_l^{\pm} = \{P \in \Gamma : \cos(x_l, n) = \pm 1\}, \qquad \omega_l^{\pm} = \omega \cup \Gamma_l^{\pm}, \quad l = 1, 2,
\end{aligned}$$

where n is the unit outer normal to the boundary $\partial\Omega$.