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An efficient and highly accurate solver for multi-body acoustic scattering problems involving rotationally symmetric scatterers

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ABSTRACT

A numerical method for solving the equations modeling acoustic scattering in three dimensions is presented. The method is capable of handling several dozen scatterers, each of which is several wave-lengths long, on a personal work station. Even for geometries involving cavities, solutions accurate to seven digits or better were obtained. The method relies on a Boundary Integral Equation formulation of the scattering problem, discretized using a high-order accurate Nyström method. A hybrid iterative/direct solver is used in which a local scattering matrix for each body is computed, and then GMRES, accelerated by the Fast Multipole Method, is used to handle reflections between the scatterers. The main limitation of the method described is that it currently applies only to scattering bodies that are rotationally symmetric.

1. Introduction

This paper presents a robust and highly accurate numerical method for modeling frequency domain acoustic scattering on a domain external to a group of scatterers in three dimensions. The solver is designed for the special case where each scatterer is rotationally symmetric, and relies on a Boundary Integral Equation (BIE) formulation of the scattering problem.

The contribution of this paper is to combine several recently developed techniques to obtain a solver capable of solving scattering problems on complex multibody geometries in three dimensions to seven digits of accuracy or more. In particular, the solver is capable of resolving domains involving cavities such as, e.g., the geometry shown in Fig. 5(a).

The solution technique proposed involves the following steps:

(1) *Reformulation*. The problem is written mathematically as a BIE on the surface of the scattering bodies using the "combined field" formulation [1,2]. See Section 2 for details.
(2) *Discretization*. The BIE is discretized using the Nyström method based on a high-order accurate composite Gaussian quadrature rule. Despite the fact that the kernel in the BIE is singular, high accuracy can be maintained using the correction techniques of [3,4]. Following [5], we exploit the rotational symmetry of each body to decouple the local equations as a sequence of equations defined on a generating contour [6–10]. This dimension reduction technique requires an efficient method for evaluating the fundamental solution of the Helmholtz equation in cylindrical coordinates (the so called "toroidal harmonics"); we use the technique described in [11]. See Section 3 for details.

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(3) *Iterative solver*. The dense linear system resulting from the Nyström discretization of the BIE is solved using the iterative solver GMRES [12], combined with a block-diagonal pre-conditioner, as in, e.g., [13, Section 6.4]. This pre-conditioner exploits that a highly accurate discrete approximation to the scattering matrix for each individual scatterer can be computed efficiently. See Section 4 for details.
(4) *Fast matrix–vector multiplication*. The application of the coefficient matrix in the iterative solver is accelerated using the Fast Multipole Method (FMM) [14], specifically the version for the Helmholtz equation developed by Gimbutas and Greengard [15].
(5) *Skeletonization*. In situations where the individual scatterers are not packed very tightly, the number of degrees of freedom in the global system can be greatly reduced by exploiting rank deficiencies in the off-diagonal blocks of the coefficient matrix. Specifically, we use a variation of the scheme introduced in [16], and further developed in [17]. Randomized methods are used to accelerate the computation of low-rank approximations to large dense matrices [18]. See Section 5 for details.

The present work draws on several recent papers describing techniques for multibody scattering, including [13], which applies a very similar technique to acoustic scattering in two dimensions. [19] addresses the harder problem of electromagnetic scattering in 3D (as opposed to the acoustic scattering considered here), but uses classical scattering matrices expressed in spherical harmonics. This is a more restrictive framework than the one used in [13] for problems in 2D, and in the present work for problems in 3D. The more general model for a compressed scattering matrix that we use here allows for larger scatterers to be handled, and also permits it to handle scatterers closely packed together. For a deeper discussion of different ways of representing compressed scattering matrices, see [20].

To describe the asymptotic cost of the method presented, let m denote the number of scatterers, let n denote the total number of discretization nodes on a single scatterer and let I denote the number of iterations required in our pre-conditioned iterative solver to achieve convergence. The cost of building all local scattering matrices is then $O(mn^2)$, and the cost of solving the linear system consists of the time T_{FMM} required for applying the coefficient matrices using the FMM, and the time T_{precond} required for applying the block-diagonal preconditioner. These scale as $T_{\text{FMM}} \sim Imn$ and $T_{\text{precond}} \sim Imn^{3/2}$ (cf. Remark 4), but for practical problem sizes, the execution time is completely dominated by the FMM. For this reason, we implemented a "skeletonization" compression scheme [16] that reduces the cost of executing the FMM from Imn to Imk, where k is a numerically determined "rank of interaction". We provide numerical examples in Section 6 that demonstrate that when the scatterers are moderately well separated, k can be smaller than n by one or two orders of magnitude, leading to dramatic practical acceleration.

2. Mathematical formulation of the scattering problem

Let $\{\Gamma_p\}_{p=1}^m$ denote a collection of m smooth, disjoint, rotationally symmetric surfaces in $\mathbb{R}^3$, let $\Gamma = \cup_{p=1}^m \Gamma_p$ denote their union, and let Ω denote the domain exterior to Γ. Our task is to compute the "scattered field" u generated by an incident field v that hits the scattering surface Γ, see Fig. 1. For concreteness, we consider the so called "sound-soft" scattering problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2 u(\boldsymbol{x}) = 0 & \boldsymbol{x} \in \Omega^{\mathrm{c}}, \\ u(\boldsymbol{x}) = -v(\boldsymbol{x}) & \boldsymbol{x} \in \Gamma, \\ \dfrac{\partial u(\boldsymbol{x})}{\partial r} - i\kappa u(\boldsymbol{x}) = O(1/r) & r := |\boldsymbol{x}| \to \infty. \end{cases} \tag{1}$$

We assume that the "wave number" κ is a real non-negative number. It is known [1] that (1) has a unique solution for every incoming field v.

Following standard practice, we reformulate (1) as second kind Fredholm Boundary Integral Equation (BIE) using a so called "combined field technique" [1,2]. We then look for a solution u of the form

$$u(\boldsymbol{x}) = \int_\Gamma G_\kappa(\boldsymbol{x}, \boldsymbol{x}')\, \sigma(\boldsymbol{x}')\, dA(\boldsymbol{x}'), \quad \boldsymbol{x} \in \Omega^c, \tag{2}$$

where G_κ is a combination of the single and double layer kernels,

$$G_\kappa(\boldsymbol{x}, \boldsymbol{x}') = \frac{\partial \phi_\kappa(\boldsymbol{x}, \boldsymbol{x}')}{\partial \boldsymbol{n}(\boldsymbol{x}')} + i\kappa\, \phi_\kappa(\boldsymbol{x}, \boldsymbol{x}') \tag{3}$$

and where ϕ_κ is the free space fundamental solution

$$\phi_\kappa(\boldsymbol{x}, \boldsymbol{x}') = \frac{e^{i\kappa|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\,|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{4}$$

Eq. (2) introduces a new unknown function σ, which we refer to as a "boundary charge distribution". To obtain an equation for σ, we take the limit in (2) as $\boldsymbol{x}$ approaches the boundary Γ, and find that σ must satisfy the integral equation

$$\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma G_\kappa(\boldsymbol{x}, \boldsymbol{x}')\, \sigma(\boldsymbol{x}')\, dA(\boldsymbol{x}') = -v(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma. \tag{5}$$