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The Fourier finite element method for the corner singularity expansion of the Heat equation

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ABSTRACT

Near the non-convex vertex the solution of the Heat equation is of the form $u = (c \star \mathcal{E})\chi\, r^{\pi/\omega}\sin(\frac{\pi\theta}{\omega}) + w$, $w \in L^2(\mathbb{R}^+; H^2)$, where c is the stress intensity function of the time variable t, $\star$ the convolution, $\mathcal{E}(\mathbf{x}, t) = re^{-r^2/4t}/2\sqrt{\pi t^3}$, χ a cutoff function and ω the opening angle of the vertex. In this paper we use the Fourier finite element method for approximating the stress intensity function c and the regular part w, and derive the error estimates depending on the regularities of c and w. We give some numerical examples, confirming the derived convergence rates.

1. Introduction

The purpose of this paper is to apply the Fourier finite element method (FFEM) to the corner singularity expansion for the Heat equation on non-convex polygonal domains, to show unique existence and error estimates, and to confirm the derived convergence rates by numerical experiments. The FFEM combines the Fourier method with the finite element method. This provides some advantages. The approximate solution of the Heat equation on polygonal domains or the boundary value problem on axisymmetric domains can be reduced to the approximate solution of a finite set of boundary value problems in two dimensions and can be solved in parallel. Also, the approximate stress intensity function can be calculated by truncated Fourier series, with coefficients of singular functions in two dimensions (see [1]).

The FFEM is based on the method given in Ref. [2], where it was applied to a general second-order elliptic Dirichlet boundary value problem on axisymmetric domains in $\mathbb{R}^3$. In [1] it was also applied to the Dirichlet problem of the Poisson equation in axisymmetric domains with reentrant edges and in [3] the interface problem of the Poisson-like equation in axisymmetric domains with edges. In [4,5] the Fourier singular complement method was introduced and analyzed in such axisymmetric domains. In [6] a combination of the FFEM with the Nitsche finite element method was applied to the Dirichlet problem of the Poisson equation in 3D axisymmetric domains with non-axisymmetric data. In particular, compared with this paper, the analysis given in [7] is similar in applying the FFEM to the edge singularity expansion of the Poisson problem but, in this paper, the stress intensity factor of the corner singularity for the Poisson problem with parameter is differently formulated (see Remark 1.3).

This paper provides a FFEM to overcome corner singularities for the Heat equation with (nearly) optimal order convergence. For the optimality we try to find the approximation of the regular part in the corner singularity expansion before computing the stress intensity function of the singular solution (cf. [7]). Even though our approach needs the knowledge of the exact forms of corner singularities, its advantage is that there is no procedure refining triangulations near the non-convex corner, compared with Ref. [8].

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Handling singularity caused by singular boundaries is a very important issue in mechanics and numerical computations because, as expected, the corners and edges of boundary may result in mechanically singular forces and numerically unstable phenomena. The finite element methods to overcome the corner singularities of the second order elliptic boundary value problem were investigated in [9–15]. Even the singularities of the parabolic problem were studied so some numerical methods on domains with corners were analyzed in [8,16–18]. In this paper we also consider the initial and boundary value problems for the Heat equation:

$$\begin{aligned} &\partial_t u - \Delta u = f \quad \text{in } \Omega \times \mathbb{R}^+, \\ &u = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ &u(\cdot, 0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where u is the unknown function and f a given function; $\Omega \subset \mathbb{R}^2$ is a non-convex polygon with the boundary $\Gamma := \partial\Omega$; $\mathbb{R}^+ := (0, \infty)$ is the positive real line.

It is assumed, for simplicity, that Ω has only one non-convex vertex P placed at the origin. Let $\omega = \omega_2 - \omega_1 > \pi$ be the opening angle of P where ω_i are the numbers in $\omega_1 < \omega_2 < \omega_1 + 2\pi$. Set $\alpha = \pi/\omega$. Denote by $r > 0$ the radial coordinate and $\theta \in (\omega_1, \omega_2)$ the angular coordinate. The corresponding corner singularity ϕ and its dual function ψ are given by

$$\phi = \chi_1 r^{\alpha} \sin[\alpha(\theta - \omega_1)], \qquad \psi = \chi_2 r^{-\alpha} \sin[\alpha(\theta - \omega_1)], \tag{1.2}$$

where $\chi_i \in C^{\infty}(\mathbb{R}^2)$ are the cutoff functions defined by

$$\chi_j = 1 \quad \text{for } r \le d_j \text{ and } 0 \text{ for } r \ge d_{j+1}, \tag{1.3}$$

where d_j are numbers with $0 < d_1 < d_2 < d_3 \ll 1$.

The spaces and norms used in this paper are as follows. For real s, H^s means the usual fractional order Sobolev space with norm $\|v\|_s$ (see [19,20]). We write $\mathrm{L}^2 = \mathrm{H}^0$ with norm $\|v\|_0 = (\int_\Omega |v|^2 dx)^{1/2}$, $\mathrm{H}^1_0 := \{v \in \mathrm{H}^1 : v|_\Gamma = 0\}$ and $\mathrm{H}^s_0 = \mathrm{H}^s \cap \mathrm{H}^1_0$. Also H^{-s} means the dual space of H^s_0 with norm

$$\|f\|_{-s} := \sup_{0 \ne v \in \mathrm{H}^s_0} \langle f, v\rangle / \|v\|_s, \tag{1.4}$$

where $\langle\, ,\, \rangle$ denotes the duality pairing. The function $u(\mathbf{x}, t)$ is considered as a mapping $u : \mathbb{R}^+ \mapsto X$ defined by $[u(t)](\mathbf{x}) := u(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$, where X is a Banach space with norm $\|\cdot\|$. Let $\mathrm{L}^2(\mathbb{R}^+; X)$ be the set of all measurable functions with

$$\|u\|_{\mathrm{L}^2(\mathbb{R}^+;X)} := \left(\int_0^\infty \|u(t)\|^2 dt\right)^{1/2}.$$

Throughout this paper C denotes a generic positive constant, for instance, $C = C(\Omega, \ldots)$, depending on Ω and so on.

The solution of the elliptic boundary value problems on a polygonal domain, for instance, the Poisson problem: $-\Delta u = f$ in Ω and $u = 0$ on Γ can be written in the following form near the non-convex vertex (see [20–23]):

$$u = \mathfrak{C}\phi + w, \quad w \in \mathrm{H}^2$$

with the regularity estimate: $\|w\|_2 + |\mathfrak{C}| \le C\|f\|_0$ for a constant C. In the numerical analysis a main issue is how the (optimally) convergent numerical solutions for the pair $[\mathfrak{C}, w]$ can be constructed. Such investigation can be found in the following references: [11–15]. In [11] the extraction formula for the coefficient $\mathfrak{C}$ is presented, based on the dual singular function method, and the error estimates are derived: $|\mathfrak{C} - \mathfrak{C}_h| + \|w - w_h\|_0 = O(h^{1+\alpha-\epsilon})$ for $0 < \epsilon \ll 1$ and $\|w - w_h\|_1 = O(h)$ where w_h and $\mathfrak{C}_h$ are the approximations of w and $\mathfrak{C}$ respectively. In [13] the multi-grid methods for the computation of singular solutions and stress intensity factors are studied and the error estimates are derived: $|\mathfrak{C} - \mathfrak{C}_h| = O(h^{1+\alpha-\epsilon})$, $\|w - w_h\|_1 = O(h)$. In [14,15] the extraction formula given in [11] is modified by the expression containing only the regular part w and the discrete variable w_h is computed by using the Sherman–Morrison formula and also the error estimates: $|\mathfrak{C} - \mathfrak{C}_h| + \|w - w_h\|_0 = O(h^{1+\alpha-\epsilon})$, $\|w - w_h\|_1 = O(h)$ are shown. Furthermore, some noticeable works on the finite element methods for elliptic boundary value problems on domains with cusps can be found in [24,25].

On the other hand such numerical analysis for the corner singularity decomposition of the Heat equation has not been given yet. A direct numerical approach to the solution itself can be found in the references: [8,16–18]. In [8] the authors show that the approximation u_h for the semidiscrete formulation of the problem (1.1) satisfies the error estimates: $\|u(t) - u_h(t)\|_0 = O(h^{2\alpha})$, $\|\nabla(u(t) - u_h(t))\|_0 = O(h^{\alpha})$ and also the optimal order convergence rates can be restored by systematically refining triangulations toward the non-convex corner.

Unlike the corner singularity expansion of the Poisson problem the corner singularity of a non-convex vertex for the time-dependent problem (1.1) corresponds to each time $t > 0$ and is of the form (Theorem 1.1)

$$u = (\mathcal{E} \star c)\phi + w, \tag{1.5}$$

where $\star$ is the convolution in time, ϕ is the corner singularity in (1.2) and w is the smoother part. We here state the regularity result for the Heat equation (1.1) on the non-convex polygon Ω (see [26, Theorem 2.2]).