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Generalized Schultz iterative methods for the computation of outer inverses

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ABSTRACT

We consider a general matrix iterative method of the type $X_{k+1} = X_k p(AX_k)$ for computing an outer inverse $A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$, for given matrices $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$ such that $A\mathcal{R}(G) \oplus \mathcal{N}(G) = \mathbb{C}^m$. Here $p(x)$ is an arbitrary polynomial of degree d. The convergence of the method is proven under certain necessary conditions and the characterization of all methods having order r is given. The obtained results provide a direct generalization of all known iterative methods of the same type. Moreover, we introduce one new method and show a procedure how to improve the convergence order of existing methods. This procedure is demonstrated on one concrete method and the improvement is confirmed by numerical examples.

1. Introduction

Generalized inverse computation is an important problem for many practical applications, including statistics and automatic control. They are defined as suitable generalizations of the ordinary inverse for a non-singular matrix $A \in \mathbb{C}^{m\times n}$. Consider the following matrix equations (known as Penrose equations [1,2]):

$$(1)\ AXA = A,\qquad (2)\ XAX = X,\qquad (3)\ (AX)^* = AX,\qquad (4)\ (XA)^* = XA.$$

Any matrix $X \in \mathbb{C}^{n\times m}$ which satisfies Eq. (2), i.e. $XAX = X$, is called an *outer* inverse of A and denoted by $X = A^{(2)}$. If we require that X is the solution of all equations (1)–(4) then such matrix $X = A^{\dagger}$ is unique and called the *Moore–Penrose inverse*. There are also a few more types of generalized inverses satisfying some of the following additional equations

$$(1^l)\ A^lXA = A^l,\qquad (5)\ AX = XA,\qquad (3M)\ (MAX)^* = MAX,\qquad (4N)\ (XAN)^* = XAN.$$

The unique matrix $X = A^D \in \mathbb{C}^{n\times n}$ satisfying (1^l), (2) and (5), for a given square matrix $A \in \mathbb{C}^{n\times n}$ is called the *Drazin inverse* of A. Here, $l = \mathrm{ind}(A)$ is the smallest integer such that $\mathrm{rank}(A^l) = \mathrm{rank}(A^{l+1})$. On the other side, matrix $X = A^{\dagger}_{M,N} \in \mathbb{C}^{n\times m}$, satisfying (1), (2), $(3M)$ and $(4N)$ for the given Hermitian symmetric and positive definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$, is called the *weighted Moore–Penrose inverse* of A. For more information regarding basic and advanced properties of the generalized inverse, see for example [1,2].

If T is a subspace of $\mathbb{C}^n$ and S is a subspace of $\mathbb{C}^m$ such that

$$AT \oplus S = \mathbb{C}^m$$

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then there exists a unique matrix $X \in \mathbb{C}^{n\times m}$ such that $XAX = X$, $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$. This matrix is called an outer inverse with a prescribed range and null space and denoted by $X = A^{(2)}_{T,S}$. If $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$ for $G \in \mathbb{C}^{n\times m}$, then X is an outer inverse corresponding to the matrix G. We will refer to X as the *G-inverse*.

It is well-known [1] that X reduces to various generalized inverses, by an appropriate choice of G. Hence

$$X = \begin{cases} A^{-1}, & G = A^*,\ A \text{ is regular;} \\ A^{\dagger}, & G = A^*; \\ A^{\dagger}_{M,N}, & G = A^{\sharp}; \\ A^{\#}, & G = A,\ \mathrm{ind}(A) = 1; \\ A^{D}, & G = A^{l},\ l \geq \mathrm{ind}(A); \end{cases} \tag{1.1}$$

where $A^{\sharp} = N^{-1}A^*M$ denotes the weighted conjugate-transpose matrix of A. There are many papers which deal with the representation, approximation and computation of outer G-inverses (see for example [3–7]). If $s = \mathrm{rank}(G) = \mathrm{rank}(AG)$ and $G = VW$ is the full-rank factorization, where $V \in \mathbb{C}^{n\times s}$ and $W \in \mathbb{C}^{s\times m}$, the inverse X can be computed by [6]:

$$X = V(WAV)^{-1}W.$$

Note that $\mathcal{R}(X) = \mathcal{R}(V)$ and $\mathcal{N}(X) = \mathcal{N}(W)$. All methods for computing (generalized) matrix inverses are divided into two major classes: *direct* and *iterative* methods. Recently published direct methods are usually based on SVD (Singular Value Decomposition), *QR* factorization [8], Cholesky (or LDL^*) factorization [9,10], Gaussian elimination [6,11], etc.

There are also a number of iterative methods for the computation of the inverse matrix or some of its generalized inverses. The most important is the Schultz method given by

$$X_{k+1} = X_k(2I - AX_k). \tag{1.2}$$

It is shown that the Schultz method can be used for the Moore–Penrose [12], Drazin [13] and G-inverse [14,15] computation. The initial matrix is chosen by $X_0 = \alpha G$ such that $|1 - \alpha\lambda_i| < 1$ where λ_i $(i = 1, 2, \ldots, s)$ are non-zero eigenvalues of AG and $s = \mathrm{rank}(AG) = \mathrm{rank}(G)$.

One can define a natural generalization of the Schultz method by:

$$X_{k+1} = X_k(I + R_k + \cdots + R_k^{r-1}), \quad R_k = I - AX_k. \tag{1.3}$$

This is a well-known method (called the *hyper-power method*) for an inverse matrix computation and has the order r. It is extended to the computation of the (1) inverse [16], Moore–Penrose inverse [17,18] and G-inverse [19,20]. The method (1.3) can be also written in the form:

$$X_{k+1} = X_k p(AX_k), \qquad p(x) = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} x^{i-1}.$$

Many different methods of the same type $X_{k+1} = X_k p(AX_k)$ appeared in the past few years (see for example [21] and later [22–25]). We refer to them as the *generalized Schultz* methods. Those methods are usually induced by applying a non-linear equation solver to the equation $F(x) = x^{-1} - a = 0$, while methods from [21,25] are based on the Penrose equations (2) and (4).

Some other variants of iterative methods for computing generalized inverses are given in [26]. All mentioned papers consider the particular treatment of those methods and the proof of convergence for some particular generalized inverse.

The aim of this paper is to show the general result regarding the convergence of the arbitrary generalized Schultz method in the case of the outer G-inverse. As a particular result, we obtain extensions of the previously mentioned methods. Furthermore, we show the characterization of all polynomials $p(x)$ such that the generalized Schultz method $X_{k+1} = X_k p(AX_k)$ has the order of convergence at least r. Such characterization provides the mechanism to construct many new methods of the same kind. This mechanism is demonstrated by improving the order of convergence of one known method and by constructing one new method. Both new methods and a few existing methods are tested on different randomly generated matrices.

2. Generalized Schultz iterative method for computing a^{-1}

Consider the following scalar iterative method

$$x_{k+1} = x_k p(ax_k) \tag{2.1}$$

for solving the equation $ax = 1$ where $a \neq 0$ is an arbitrary (complex) number and

$$p(x) = p_0 + p_1 x + \cdots + p_d x^d \tag{2.2}$$

is a given polynomial of degree d. Method of type (2.1) usually appears by applying some non-linear equation solver to the equation $f(x) = x^{-1} - a = 0$ (see for example [22–24]).

Proposition 2.1. *Assuming that method (2.1) is convergent, i.e. that $x_k \to 1/a$ when $k \to +\infty$, then $p(1) = 1$.*